Non-supersymmetric microstates of the D1-D5-KK system

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Abstract

We construct a discrete family of smooth non-supersymmetric three charge geometries carrying D1 brane, D5 brane and Kaluza-Klein monopole charges in Type IIB supergravity compactified on a six-torus, which can be interpreted as the geometric description of some special states of the brane system. These solutions are asymptotically flat in four dimensions, and generalise previous supersymmetric solutions. The solutions have a qualitatively similar structure to previous non-supersymmetric smooth solutions carrying D1 and D5 brane charges in five dimensions, and indeed can be viewed as the five-dimensional system placed at the core of a Kaluza-Klein monopole. The geometries are smooth, free of horizons and do not have closed timelike curves. One notable difference from the five-dimensional case is that the four-dimensional geometry has no ergoregion.

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1 Introduction

An important goal for the study of black hole thermodynamics is to understand the gravitational description of the microstates responsible for the entropy. String theory and in particular the AdS/CFT correspondence offers the tools needed to explore these issues. The past few years have seen significant progress in our understanding of the geometrical description of the states underlying some special black holes which can be embedded in string theory. Though a generic microstate responsible for the black hole entropy is expected to admit a description only in the full string theory, there is at least a subset of these states which can be well described by supergravity solutions. Probably the best studied example is the supersymmetric black hole in five dimensions [1, 2]. The microstates of the five dimensional black hole with two charges, which has a string-scale horizon if we take into account higher derivative corrections, have been completely described [3]-[12]. For the black hole with three charges in five dimensions, which has a macroscopic horizon, many explicit examples of the microstates are known [13]-[24], though the picture is far less complete. Similar results have been achieved for the case of three and four charge systems in four dimensions [25]-[27]. For a review of some of these developments, see [28, 29].

The results mentioned above refer to systems with unbroken supersymmetry in four or five dimensions. It is an important and non-trivial task to extend the success of the supersymmetric case to the more general non-supersymmetric states. Although the supersymmetric black holes already have finite horizon areas, the non-supersymmetric ones are qualitatively different: notably, because they have a non-zero temperature. This implies that the study of non-supersymmetric black holes is significantly more complex; it will involve issues like Hawking radiation and dynamical instabilities. Also, from a technical point of view, the task of finding supersymmetric microstates is greatly facilitated by the classification theorems in supergravity which hold in the presence of some unbroken supersymmetry [30]. For the non-supersymmetric case, these techniques are not available.

The only known geometries describing non-supersymmetric microstates are the ones of [31]. In the Type IIB duality frame, these solutions carry D1, D5 and momentum charges in five dimensions. A natural problem is to extend these solutions by adding a Kaluza-Klein (KK) monopole charge to the system, to produce non-supersymmetric microstates of the four-charge system in four dimensions. In the supersymmetric case, the analogous problem can be solved in a systematic manner. The results of [30] imply that a large class of supersymmetric solutions can be described by a set of harmonic functions. In this language, adding KK monopole charge turns out to be equivalent to adding appropriate constants to some of these harmonic functions. However, the analysis of [32] has shown that the linear structure underlying the supersymmetric solutions is completely destroyed when we pass to the non-supersymmetric case. Thus, the solution of this problem will require the use of different techniques. We will approach this problem by the same route taken to construct the five-dimensional non-supersymmetric microstates

in [31]. We will first construct a suitable general family of stationary geometries, and then find constraints on the parameters to obtain smooth solutions.

Qualitatively, we would expect the relevant solutions to look like the five-dimensional solutions of [31] placed at the core of a KK monopole. We could attempt to directly add the KK monopole charge to the general metric considered in [31], which was first obtained in [33]. However, adding the KK monopole charge to the charged solution would be quite complicated. Instead, we observe that solutions with D1, D5 and momentum charges can be obtained by starting from a suitable vacuum solution and applying a sequence of boosts and dualities. For the solution of [33], the relevant vacuum solution was the Myers-Perry black hole. We can add the KK monopole charge to this vacuum "seed" solution, and then subsequently add the other charges. This is a useful way to proceed because there are powerful solution-generating transformations for the vacuum solutions, based on an $SL(3,\mathbb{R})$ symmetry of the equations of motion [34]. This solution-generating transformation was used to construct black hole solutions with KK electric and magnetic charges in [35, 36]. It has recently been shown that it can be used to add KK monopole charge to any stationary, axisymmetric solution of the vacuum equations [37, 38]. The black hole solutions of [35, 36] might appear at first glance to provide appropriate "seeds" for us, but they correspond only to under-rotating versions of the Myers-Perry black hole placed at the core of the KK monopole, while smooth solutions are obtained by considering over-rotating black holes. In section 2, we therefore construct new seeds, starting from the Kerr-Bolt instanton. Once we add the KK monopole charge, the solutions we obtain will turn out to be an analytic continuation (in parameter space) of the solutions of [35, 36], and they indeed describe an over-rotating Myers-Perry black hole at the core of the KK monopole. The general solution carries KK electric and magnetic charges and angular momentum in four dimensions. The KK electric charge and angular momentum in four dimensions correspond to the two independent angular momenta in five dimensions, so we would expect them to be determined in terms of the other conserved charges when we obtain a smooth solution.

Once we have obtained appropriate vacuum "seed" solutions, we add D1 and D5 charges by a sequence of boosts and dualities in section 3. In this paper we restrict the analysis to the case with zero momentum charge. The general case has some additional complications which will be studied in a forthcoming publication. The solution is given in section 3.3; the reader not interested in the details of its construction can skip to this point.

In section 4, we identify solutions corresponding to microstates of the brane system by a systematic search of the parameter space for values at which all the singularities can be removed. We find that as expected, the smooth solutions are determined by the D1, D5 and KK monopole charges, and an integer $n \geq 1$. For all values of n greater than 1 the solutions are non-supersymmetric; for $n = 1$ the solution reduces to the supersymmetric D1-D5-KK microstate found in [25]. In section 5, we verify that the solutions identified in section 4 are free of horizons, curvature singularities and closed time-like curves, and that the matter fields are also regular.

In section 6, we study some properties of the solitons. We find that there is a limit in which the solutions have a near-core geometry which is an orbifold of $AdS_3 \times S^3$; as in [31], obtaining this limit requires a suitable scaling of the charges. Thus these solutions are good candidates to describe microstates of the D1-D5-KK black hole. A rather surprising feature of these solutions is that in the four-dimensional metric, there is no ergoregion. This is in contrast to the five-dimensional solutions of [31], where all the non-supersymmetric solutions had an ergoregion. This implies that the instability identified for the five-dimensional solutions in [39] will not appear for these four-dimensional solutions. Investigating their stability is an important open problem. We also show that if we write the four-dimensional solutions as a fibration over a three-dimensional base space, this base space is identical to that obtained for the five-dimensional solutions of [31] in [32]. Hence, as argued in [32], the picture of four-dimensional solutions as built up out of half-BPS "atoms" of [27] does not apply to these non-supersymmetric solutions.

In the future, we would like to extend this class of solutions by adding momentum charge, thus producing non-supersymmetric microstates of the four charge black hole. This is not as straightforward as one might imagine, because the three charge solutions constructed here also carry an induced KK monopole charge along the y direction. Adding momentum along y by boosting in that direction will therefore produces NUT charges in the solution, which makes it asymptotically not flat (in four dimensions). It might be possible to cancel this NUT charge by starting with a seed solution which already carries some NUT charge. Then one can attempt to cancel the induced NUT charge against the one present in the seed metric. The details of the construction, however, are likely to be complicated.

Another important issue to address is the stability of these solitons. It was shown in [39] that the five dimensional non-supersymmetric microstates of [31] suffer from a classical instability which arises from the presence of an ergoregion. We have shown that the four dimensional geometries we construct here do not have a four-dimensional ergoregion, so we expect that they do not suffer from this particular type of instability. It would be very interesting to investigate other possible instabilities of this system.

It would also be interesting to relate the geometric picture of the microstates found here to a microscopic description. It would be particularly interesting to consider the behaviour of these microstates as we vary the coupling, along the lines of [27], and see if they can be related to some quiver gauge theory description at weak coupling.

2 Over-rotating vacuum solution

We begin by constructing a suitable vacuum solution carrying KK electric and magnetic charges. As explained in the introduction, it is easier to add the KK monopole charge to the vacuum solution and then add the D1 and D5 charges, because we can add KK monopole charge to any five dimensional stationary axisymmetric vacuum solution of Einstein equation by an $SL(3,\mathbb{R})$ solution-generating transformation [37, 38]. The resulting general solution will also carry a KK electric charge; this can be thought of as

corresponding to angular momentum along the fiber direction in the five-dimensional geometry. We need to construct new vacuum solutions because the known black hole solutions of [35, 36] only describe under-rotating black holes. On the other hand, in order to construct microstates one needs a family of solutions containing horizon-free geometries. We could construct appropriate solutions by applying the procedure of [37, 38] to the over-rotating Myers-Perry solution. One finds, however, that these solutions lie in the same $SL(3,\mathbb{R})$-orbit as the Kerr-Bolt instanton trivially lifted to five dimensions. Hence one can equivalently construct the required vacuum solution by applying an SO(2,1) transformation to the Kerr-Bolt instanton. This construction has the advantage of providing a parametrization which is similar to the one used in [35, 36] and, in fact, the solution we obtain is related to the one of [35, 36] by a simple analytic continuation in parameter space.

2.1 The solution generating technique

Let us briefly review the solution generating technique of [34]. A stationary solution of five-dimensional Einstein equations can be brought to the form

$$ds_5^2 = g_{ab}(d\xi^a + \omega^a{}_i dx^i)(d\xi^b + \omega^b{}_j dx^j) + \frac{1}{\tau} ds_3^2\,, \quad \tau = -\det g_{ab}, \tag{2.1}$$

where $a, b = 0, 1$ and $\xi^0 \equiv t$, $\xi^1 \equiv z$. z is a compact coordinate and $\frac{\partial}{\partial z}$ is assumed to be Killing. ω^a are gauge fields on the three-dimensional space parametrized by x^i, and thus they can be dualized to scalars, V_a, such that

$$dV_a = -\tau g_{ab} *_3 d\omega^a, \tag{2.2}$$

where $*_3$ is performed with the metric ds_3^2. Introduce the 3×3 unimodular matrix

$$\chi = \begin{pmatrix} g_{ab} - \frac{1}{\tau} V_a V_b & \frac{1}{\tau} V_a \\ \frac{1}{\tau} V_b & -\frac{1}{\tau} \end{pmatrix}. \tag{2.3}$$

The equations of motion can be written as

$$d *_3 (\chi^{-1} d\chi) = 0 \tag{2.4}$$

and

$$R^{(3)}_{ij} = \frac{1}{4} \mathrm{Tr}(\chi^{-1} \partial_i \chi\, \chi^{-1} \partial_j \chi). \tag{2.5}$$

As shown in [37], it is useful to interpret Eq. (2.4) as the integrability condition for the following:

$$\chi^{-1} d\chi = *_3 d\kappa. \tag{2.6}$$

This defines a 3×3 matrix of 1-forms κ. One has that

$$\omega^0 = -\kappa^0{}_2\,, \quad \omega^1 = -\kappa^1{}_2. \tag{2.7}$$

The equations of motion are invariant under the linear transformation

$$\chi \to N\chi N^T \ , \ \kappa \to (N^T)^{-1}\kappa N^T \ , \quad N \in \mathrm{SL}(3,\mathbb{R}) \tag{2.8}$$

if the base metric ds_3^2 is kept fixed. This $\mathrm{SL}(3,\mathbb{R})$ group of transformations can be used to generate new solutions from known ones. If one wants to preserve the asymptotic structure of the solution, which in our case is $\mathbb{R}^{3,1} \times S^1$, the transformation matrix N has to be restricted to the subgroup $\mathrm{SO}(2,1)$.

2.2 Constructing the vacuum seed metric

We want to construct a vacuum solution with the following properties: it goes asymptotically to $\mathbb{R}^{3,1} \times S^1$, it carries KK electric and magnetic charges along the S^1 and, when the size of the KK monopole is made much larger than any other length scale, the solution reduces to the *over-rotating* Myers-Perry solution. We will obtain such a solution by applying an $\mathrm{SO}(2,1)$ transformation to the following starting metric:

$$\begin{aligned} ds_5^2 &= -dt^2 + \frac{\tilde{F}}{\rho^2 - (m - b\cos\theta)^2}\left(dz - \frac{2m\tilde{\Delta}(m - b\cos\theta)}{b\tilde{F}}d\phi\right)^2 \\ &\quad +(\rho^2 - (m - b\cos\theta)^2)\left[\frac{d\rho^2}{\tilde{\Delta}} + d\theta^2 + \frac{\tilde{\Delta}}{\tilde{F}}\sin^2\theta d\phi^2\right], \end{aligned} \tag{2.9}$$

where $\tilde{F}$ and $\tilde{\Delta}$ are

$$\tilde{F} = \rho^2 + m^2 - b^2\cos^2\theta, \quad \tilde{\Delta} = \rho^2 + m^2 - b^2. \tag{2.10}$$

This is a Kerr-Bolt instanton lifted to five dimensions by adding a flat time direction. The χ and κ matrices associated to the metric (2.9) are

$$\chi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{\rho^2 - (m + b\cos\theta)^2}{\tilde{F}} & -\frac{2m\rho}{\tilde{F}} \\ 0 & -\frac{2m\rho}{\tilde{F}} & -\frac{\rho^2 - (m - b\cos\theta)^2}{\tilde{F}} \end{pmatrix}, \tag{2.11}$$

$$\kappa = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2mb\rho\sin^2\theta}{\tilde{F}} & \frac{2m\tilde{\Delta}(m - b\cos\theta)}{b\tilde{F}} - \frac{2m^2}{b} \\ 0 & 2m\left(\cos\theta - \frac{b\sin^2\theta(m + b\cos\theta)}{\tilde{F}}\right) & -\frac{2mb\rho\sin^2\theta}{\tilde{F}} \end{pmatrix} d\phi. \tag{2.12}$$

Of particular interest to us is the asymptotic behavior of κ. This is important in determining the condition for the absence of NUT charge in the solution obtained after a general SO(2,1) rotation. We find that

$$\kappa \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2m\cos\theta \\ 0 & 2m\cos\theta & 0 \end{pmatrix} \tag{2.13}$$

for large ρ. Under a transformation N, κ transforms as in (2.8); using also the fact that $\omega^0 = -\kappa^0{}_2$, we see that the transformed solution is free of NUT charge if the $(0,2)$ component of the transformed κ vanishes at large ρ. This leads to the condition

$$N_{13}N_{32} = N_{12}N_{33}. \tag{2.14}$$

A general SO$(2,1)$ matrix can be decomposed as $N = N_3 N_2 N_1$ where

$$N_1 = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.15}$$

$$N_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\beta & \sinh\beta \\ 0 & \sinh\beta & \cosh\beta \end{pmatrix}, \tag{2.16}$$

$$N_3 = \begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix}. \tag{2.17}$$

Using this parametrization of N, we can rewrite the NUT elimination condition (2.14) as

$$\tan 2\gamma = \tanh\alpha \operatorname{csch}\beta. \tag{2.18}$$

In order to impose this condition we will find it most convenient to solve the above equation for α, leaving β and γ as free parameters. Using now the transformation rule (2.8), and reconstructing the components of the transformed metric from the transformed χ and κ, we arrive at the following metric:

$$ds_5^2 = \frac{B}{A}(dz + A_\mu dx^\mu)^2 - \frac{f^2}{B}(dt + \omega^0_\phi d\phi)^2 + A\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2\right), \tag{2.19}$$

where

$$\Delta = r^2 - 2Mr + P^2 + Q^2 - 3\Sigma^2 - b^2, \tag{2.20}$$

$$f^2 = r^2 - 2Mr - b^2\cos^2\theta + P^2 + Q^2 - 3\Sigma^2, \tag{2.21}$$

$$
\begin{aligned}
A_\mu dx^\mu &= \frac{C}{B}dt + \left(\omega^1_\phi + \frac{C}{B}\omega^0_\phi\right) d\phi, && (2.22)\\
A &= (r-\Sigma)^2 - \frac{2P^2\Sigma}{\Sigma - M} - b^2\cos^2\theta + \frac{2JPQ\cos\theta}{(M+\Sigma)^2 - Q^2}, && (2.23)\\
B &= (r+\Sigma)^2 - \frac{2Q^2\Sigma}{\Sigma + M} - b^2\cos^2\theta - \frac{2JPQ\cos\theta}{(M-\Sigma)^2 - P^2}, && (2.24)\\
C &= 2Q\,(r-\Sigma) - \frac{2PJ\cos\theta(M+\Sigma)}{(M-\Sigma)^2 - P^2}, && (2.25)\\
\omega^0_\phi &= \frac{2J\sin^2\theta}{f^2}\left[r - M + \frac{(M^2+3\Sigma^2-P^2-Q^2)(M+\Sigma)}{(M+\Sigma)^2-Q^2}\right], && (2.26)\\
\omega^1_\phi &= \frac{2P\Delta}{f^2}\cos\theta - \frac{2QJ\sin^2\theta\,[r(M-\Sigma)+M\Sigma+3\Sigma^2-P^2-Q^2]}{f^2\,[(M+\Sigma)^2-Q^2]}. && (2.27)\\
& && (2.28)
\end{aligned}
$$

We have redefined the radial coordinate as

$$\rho = r - M. \tag{2.29}$$

The constants M, Σ, Q, P and J are functions of m, b, β and γ, given by

$$
\begin{aligned}
M &= \frac{m\sinh\beta\cosh\beta}{\sqrt{1-\sin^2 2\gamma\cosh^2\beta}}, && (2.30)\\
P &= -\frac{m\cos\gamma\left(1-2\cos^2\gamma\cosh^2\beta\right)}{\sqrt{1-\sin^2 2\gamma\cosh^2\beta}}, && (2.31)\\
Q &= \frac{m\sin\gamma\left(1-2\sin^2\gamma\cosh^2\beta\right)}{\sqrt{1-\sin^2 2\gamma\cosh^2\beta}}, && (2.32)\\
J &= -\frac{mb\sin 2\gamma}{2}\left(\frac{1-\sin^2 2\gamma\cosh^4\beta}{1-\sin^2 2\gamma\cosh^2\beta}\right), && (2.33)\\
\Sigma &= -\frac{m\cos 2\gamma\sinh\beta\cosh\beta}{\sqrt{1-\sin^2 2\gamma\cosh^2\beta}}. && (2.34)
\end{aligned}
$$

It follows from this that the parameters of the solution satisfy the relations

$$
\begin{aligned}
M^2 + 3\Sigma^2 - P^2 - Q^2 + m^2 &= 0, && (2.35)\\
\frac{Q^2}{\Sigma + M} + \frac{P^2}{\Sigma - M} &= 2\Sigma, && (2.36)\\
\frac{b^2\,[(M+\Sigma)^2-Q^2]\,[(M-\Sigma)^2-P^2]}{P^2+Q^2-M^2-3\Sigma^2} &= J^2. && (2.37)
\end{aligned}
$$

In order to perform the dualities of the next subsection, we will also need the potential V_0 associated to the metric (2.19), together with the components ${\kappa^1}_{0,\phi}$ and ${\kappa^0}_{0,\phi}$ of κ.

They are given by

$$V_0 = -\frac{2}{A}(J\cos\theta + PQ), \tag{2.38}$$

$$\begin{aligned}
{\kappa^1}_{0,\phi} &= \frac{2}{f^2}\Big[Q\Delta\cos\theta \\
&\quad + \frac{JP}{(M-\Sigma)^2 - P^2}((r-M)(M+\Sigma) + P^2 + Q^2 - M^2 - 3\Sigma^2)\sin^2\theta\Big], && (2.39)\\
{\kappa^0}_{0,\phi} &= -\frac{2}{f^2}\Big[(M+\Sigma)\Delta\cos\theta + \frac{JQP}{(M-\Sigma)^2-P^2}(r-M)\sin^2\theta\Big]. && (2.40)
\end{aligned}$$

The metric (2.19) is analogous to the metric found in [35], with the crucial difference that while the metric of [35] goes over to the under-rotating Myers-Perry solution at the core of the KK monopole, the metric (2.19) approaches the over-rotating Myers-Perry solution in the same limit. As for the metric of [35], one can rewrite the solution (2.19) in a somewhat more convenient parametrization, analogous to the one found in [36]. In this new form, the parameters β and γ are exchanged for parameters p and q, defined as

$$p = M - \Sigma, \quad q = M + \Sigma. \tag{2.41}$$

The constraints (2.34) imply

$$P^2 = \frac{p(p^2+m^2)}{(p+q)}, \quad Q^2 = \frac{q(q^2+m^2)}{(p+q)}, \tag{2.42}$$

$$J^2 = b^2\frac{pq(pq-m^2)^2}{(p+q)^2 m^2}. \tag{2.43}$$

We also return to the original radial coordinate, $\rho = r - M$. Then the metric functions can be rewritten explicitly in terms of this parametrization as

$$\Delta = \rho^2 + m^2 - b^2, \tag{2.44}$$

$$f^2 = \rho^2 + m^2 - b^2\cos^2\theta, \tag{2.45}$$

$$\begin{aligned}
A &= f^2 + 2p\left[\rho + \frac{(pq-m^2)}{(p+q)} + b\frac{\sqrt{p^2+m^2}\sqrt{q^2+m^2}}{m(p+q)}\cos\theta\right], && (2.46)\\
B &= f^2 + 2q\left[\rho + \frac{(pq-m^2)}{(p+q)} - b\frac{\sqrt{p^2+m^2}\sqrt{q^2+m^2}}{m(p+q)}\cos\theta\right], && (2.47)\\
C &= 2\frac{\sqrt{q}}{\sqrt{p+q}}\left[\sqrt{q^2+m^2}(\rho+p) - \frac{q\sqrt{p^2+m^2}}{m}b\cos\theta\right], && (2.48)\\
\omega^0 &= \frac{2J\sin^2\theta}{f^2}\left[\rho - \frac{m^2(p+q)}{(pq-m^2)}\right]d\phi, && (2.49)\\
\omega^1 &= \frac{2\sqrt{p}}{\sqrt{p+q}}\frac{1}{f^2}\left[\sqrt{p^2+m^2}\Delta\cos\theta - \frac{b\sqrt{q^2+m^2}}{m}(\rho p - m^2)\sin^2\theta\right]d\phi. && (2.50)
\end{aligned}$$

The quantities needed for the dualities can be rewritten in this parametrization as

$$V_0 = -\frac{2}{A}\frac{\sqrt{pq}}{(p+q)}\left[\frac{b(pq-m^2)}{m}\cos\theta + \sqrt{p^2+m^2}\sqrt{q^2+m^2}\right], \tag{2.51}$$

$${\kappa^1}_{0,\phi} = \frac{2}{f^2}\frac{\sqrt{q}}{\sqrt{p+q}}\left[\sqrt{q^2+m^2}\Delta\cos\theta + \frac{\sqrt{p^2+m^2}b(\rho q+m^2)}{m}\sin^2\theta\right], \tag{2.52}$$

$${\kappa^0}_{0,\phi} = -\frac{2}{f^2}q\left[\Delta\cos\theta + \frac{b\sqrt{p^2+m^2}\sqrt{q^2+m^2}}{m(p+q)}\rho\sin^2\theta\right]. \tag{2.53}$$

This parametrization manifests the fact that the metric (2.19) is an analytic continuation of the metric in [36]. The two metrics are related by

$$p_L = 2p, \quad q_L = 2q, \tag{2.54}$$

$$m_L = -im, \quad a_L = ib, \tag{2.55}$$

where p_L, q_L, m_L and a_L are the parameters of [36].

3 Adding charges via dualities

In the previous subsection we have constructed a solution of the five-dimensional vacuum Einstein equations, whose asymptotic limit is $\mathbb{R}^{3,1}\times S^1$. The only charges carried by this solution are KK electric and KK magnetic charge along the S^1, which we denote by P_z and KK_z, respectively.

We can trivially lift this solution to ten dimensions by adding five flat compact directions, which we denote by y and $z_1,\ldots,z_4$. By a sequence of boosts and dualities we can add charges corresponding to D1 branes wrapped along y and D5 branes along $y, z_1,\ldots,z_4$; we denote these charges as $D1_y$ and $D5_{y1234}$. A further boost along y would add P_y charge, but we do not explicitly perform this last step in this paper. In this way we generate a non-extremal solution carrying P_z, KK_z, $D1_y$ and $D5_{y1234}$ charges. When augmented with the last P_y charge, this solution represents the most general non-extremal solution with four non-compact dimensions: all other solutions are related to this one by dualities.

Let us start by introducing some notation. We rewrite the five-dimensional vacuum solution in (2.19) as

$$ds_5^2 = -(1-H)(dt+\mathcal{A})^2 + ds_4^2, \tag{3.1}$$

where

$$(1-H) = -g_{tt}\,,\ \mathcal{A} = \omega^0 + \frac{g_{tz}}{g_{tt}}(dz+\omega^1)\,,\ g_{tt} = \frac{Af^2-C^2}{AB}\,,\ g_{tz} = \frac{C}{A},$$
$$ds_4^2 = -\frac{\tau}{g_{tt}}(dz+\omega^1)^2 + \frac{1}{\tau}ds_3^2\,,\ \tau = \frac{f^2}{A}. \tag{3.2}$$

When lifted to ten dimensions this solution becomes

$$ds^2_{10} = -(1-H)(dt+\mathcal{A})^2 + ds^2_4 + dy^2 + ds^2_{T^4}\,, \quad ds^2_{T^4} = \sum_{i=1}^{4} dz_i^2. \tag{3.3}$$

3.1 Duality chain

In the following we describe the sequence of boosts and dualities required to add the desired charges. At each step, the charges of the resulting solution will be given in parenthesis (for brevity, we will omit the starting P_z, KK_z charges, that are present throughout). Since this procedure is fairly standard by now, we will be very schematic. The only computationally challenging step is the dualization of the RR 6-form into the corresponding 2-form, so we will give more details of this step. For brevity, we introduce the notation

$$s_{1,5} = \sinh\delta_{1,5}\,, \quad c_{1,5} = \cosh\delta_{1,5}\,, \quad H_{1,5} = 1 + H\sinh^2\delta_{1,5}. \tag{3.4}$$

$B^{(2)}$ denotes the NS-NS B-field and $C^{(p)}$ the p-form RR field. Φ is the dilaton. All metrics are in string frame. Our conventions for the normalization of the gauge fields and U-duality rules are as given in Appendix A of [5].

3.1.1 Boost along y with parameter δ_5 (P_y)

The change of coordinates

$$t \to c_5 t + s_5 y\,, \quad y \to s_5 t + c_5 y \tag{3.5}$$

produces the metric

$$\begin{aligned} ds^2_{10} &= H_5\left[dy - \frac{c_5 s_5 H}{H_5}(dt + c_5\mathcal{A}) + s_5\mathcal{A}\right]^2 - \frac{(1-H)}{H_5}(dt + c_5\mathcal{A})^2 \\ &\quad + ds^2_4 + ds^2_{T^4}. \end{aligned} \tag{3.6}$$

3.1.2 T-duality along y ($F1_y$)

$$ds^2_{10} = H_5^{-1}dy^2 - \frac{(1-H)}{H_5}(dt + c_5\mathcal{A})^2 + ds^2_4 + ds^2_{T^4}, \tag{3.7}$$

$$B^{(2)} = \left[-\frac{c_5 s_5 H}{H_5}(dt + c_5\mathcal{A}) + s_5\mathcal{A}\right] \wedge dy, \tag{3.8}$$

$$e^{2\phi} = H_5^{-1}. \tag{3.9}$$

3.1.3 Boost along y with parameter δ_1 $(F1_y - P_y)$

The transformation

$$t \to c_1 t + s_1 y\,, \quad y \to s_1 t + c_1 y \tag{3.10}$$

gives

$$\begin{aligned} ds_{10}^2 &= \frac{H_1}{H_5}\left[dy - \frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) + s_1 c_5 \mathcal{A}\right]^2 \\ &\quad - \frac{(1-H)}{H_1 H_5}\left[dt + c_1 c_5 \mathcal{A}\right]^2 + ds_4^2 + ds_{T^4}^2, \qquad (3.11) \\ B^{(2)} &= -\frac{c_5 s_5 H}{H_5}\left[(dt + c_1 c_5 \mathcal{A}) \wedge (dy + s_1 c_5 \mathcal{A})\right] + s_5 \mathcal{A} \wedge (c_1 dy - s_1 dt)\,, \qquad (3.12) \\ e^{2\phi} &= H_5^{-1}. \qquad (3.13) \end{aligned}$$

3.1.4 S-duality $(D1_y - P_y)$

$$\begin{aligned} ds_{10}^2 &= \frac{H_1}{H_5^{1/2}}\left[dy - \frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) + s_1 c_5 \mathcal{A}\right]^2 \\ &\quad - \frac{(1-H)}{H_5^{1/2} H_1}\left[dt + c_1 c_5 \mathcal{A}\right]^2 + H_5^{1/2}(ds_4^2 + ds_{T^4}^2), \qquad (3.14) \\ C^{(2)} &= -\frac{c_5 s_5 H}{H_5}\left[(dt + c_1 c_5 \mathcal{A}) \wedge (dy + s_1 c_5 \mathcal{A})\right] + s_5 \mathcal{A} \wedge (c_1 dy - s_1 dt)\,, \qquad (3.15) \\ e^{2\phi} &= H_5. \qquad (3.16) \end{aligned}$$

3.1.5 T-duality along T^4 $(D5_{y1234} - P_y)$

$$\begin{aligned} ds_{10}^2 &= \frac{H_1}{H_5^{1/2}}\left[dy - \frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) + s_1 c_5 \mathcal{A}\right]^2 \\ &\quad - \frac{(1-H)}{H_5^{1/2} H_1}\left[dt + c_1 c_5 \mathcal{A}\right]^2 + H_5^{1/2} ds_4^2 + H_5^{-1/2} ds_{T^4}^2, \qquad (3.17) \\ C^{(6)} &= \left[-\frac{c_5 s_5 H}{H_5}\left[(dt + c_1 c_5 \mathcal{A}) \wedge (dy + s_1 c_5 \mathcal{A})\right] + s_5 \mathcal{A} \wedge (c_1 dy - s_1 dt)\right] \wedge dz_i^4, \qquad (3.18) \\ e^{2\phi} &= H_5^{-1}. \qquad (3.19) \end{aligned}$$

Note that the type IIB action in our conventions only includes p-forms with $p \leq 4$. Thus the 6-form gauge field generated in the step above has to be dualized to a 2-form by using the electric-magnetic duality. Note that in the general case (i.e with a non-trivial NS-NS 2-form) the duality equation is modified by the presence of Chern-Simons terms. However in the case at hand, there is no NS-NS 2-form field and the duality equations are the naive ones given below.

3.1.6 EM duality

As explained above, in order to perform the dualities that follow, we need to dualize the 6-form $C^{(6)}$ to a 2-form $C^{(2)}$. That is, we have to find a $C^{(2)}$ satisfying

$$* \, dC^{(6)} = dC^{(2)}, \tag{3.20}$$

where $*$ is performed with the metric (3.17). From (3.18) we find

$$dC^{(6)} = \Big[-\frac{c_5 s_5}{H_5^2} dH \wedge (dt + c_1 c_5 \mathcal{A}) \wedge (dy + s_1 c_5 \mathcal{A}) + \frac{(1-H)s_5}{H_5} d\mathcal{A} \wedge (c_1 dy - s_1 dt) \Big] \wedge dz_i^4. \tag{3.21}$$

Define 1-forms ω_1 and ω_2 as

$$\begin{aligned} \omega_1 &= dt + c_1 c_5 \mathcal{A}, &(3.22)\\ \omega_2 &= dy - \frac{c_1 s_1 H}{H_1}\omega_1 + s_1 c_1 \mathcal{A}, &(3.23) \end{aligned}$$

so that

$$c_1 dy - s_1 dt = c_1 \omega_2 - \frac{s_1(1-H)}{H_1}\omega_1, \tag{3.24}$$

and

$$dC^{(6)} = \left[-\frac{c_5 s_5}{H_5^2} dH \wedge \omega_1 \wedge \omega_2 + \frac{(1-H)s_5}{H_5} d\mathcal{A} \wedge \left(c_1 \omega_2 - \frac{s_1(1-H)}{H_1}\omega_1 \right) \right] \wedge dz_i^4. \tag{3.25}$$

Let $\eta^{(1)}$, $\eta^{(2)}$ be any 1 and 2-forms on ds_4^2. The Hodge star operation acts as

$$* \left[\eta^{(1)} \wedge \omega_1 \wedge \omega_2 \wedge dz_i^4 \right] = -\frac{H_5^2}{(1-H)^{1/2}} *_4 \eta^{(1)}, \tag{3.26}$$

$$* \left[\eta^{(2)} \wedge \omega_1 \wedge dz_i^4 \right] = \frac{H_1 H_5}{(1-H)^{1/2}} (*_4 \eta^{(2)}) \wedge \omega_2, \tag{3.27}$$

$$* \left[\eta^{(2)} \wedge \omega_2 \wedge dz_i^4 \right] = \frac{H_5 (1-H)^{1/2}}{H_1} (*_4 \eta^{(2)}) \wedge \omega_1. \tag{3.28}$$

We can use these relations to compute

$$\begin{aligned} * \, dC^{(6)} &= c_5 s_5 \left(\frac{*_4 dH}{(1-H)^{1/2}} + (1-H)^{3/2} (*_4 d\mathcal{A}) \wedge \mathcal{A} \right) \\ &\quad + s_5 (1-H)^{3/2} (*_4 dA) \wedge (c_1 dt - s_1 dy). &(3.29) \end{aligned}$$

The $C^{(2)}$ solving (3.20) can then be written in the form

$$C^{(2)} = c_5 s_5 \mathcal{C} + s_5 \mathcal{B} \wedge (c_1 dt - s_1 dy), \tag{3.30}$$

where the 1-form $\mathcal{B}$ and the 2-form $\mathcal{C}$ have to satisfy

$$d\mathcal{B} = (1-H)^{3/2}(*_4 d\mathcal{A}), \tag{3.31}$$

$$d\mathcal{C} = \left(\frac{*_4 dH}{(1-H)^{1/2}} + (1-H)^{3/2}(*_4 d\mathcal{A}) \wedge \mathcal{A}\right). \tag{3.32}$$

The dualization problem has thus been reduced to finding $\mathcal{B}$ and $\mathcal{C}$ that solve (3.31) and (3.32). Note that these equations involve only the seed vacuum metric.

Let us first look at $\mathcal{B}$. If we further decompose $\mathcal{B}$ as

$$\mathcal{B} = \mathcal{B}_z\,(dz + \omega^1) + \mathcal{B}_\phi\, d\phi, \tag{3.33}$$

we find that (3.31) implies

$$d\mathcal{B}_z = \tau\lambda_{0a} *_3 d\omega^a \tag{3.34}$$

and

$$d(\mathcal{B}_\phi\, d\phi) = *_3(\chi^{-1}d\chi)^1{}_0. \tag{3.35}$$

Comparing these equations with the ones defining V_a and κ, we see that

$$\mathcal{B}_z = -V_0\,, \quad \mathcal{B}_\phi\, d\phi = \kappa^1{}_0. \tag{3.36}$$

Similarly let us write

$$\mathcal{C} = (dz + \omega^1) \wedge \mathcal{C}_z, \tag{3.37}$$

where $\mathcal{C}_z$ is a 1-form on the 3D base, which in our case has only has a component along ϕ. Then (3.32) implies that

$$d(\mathcal{C}_z + V_0\omega^0) = *_3(\chi^{-1}d\chi)^0{}_0, \tag{3.38}$$

so that a solution is

$$\mathcal{C}_z = -V_0\omega^0 + \kappa^0{}_0. \tag{3.39}$$

In conclusion, we have related the solution of the duality equation (3.20) to the quantities V_0, ω^a, κ that have been computed for the 5D vacuum solution in (2.51–2.53). The RR 2-form $C^{(2)}$ dual to $C^{(6)}$ is given by (3.30) with

$$\begin{aligned} \mathcal{B} &= -V_0(dz + \omega^1) + \kappa^1{}_0, \\ \mathcal{C} &= (dz+\omega^1) \wedge [-V_0\omega^0 + \kappa^0{}_0]. \end{aligned} \tag{3.40}$$

3.1.7 S-duality ($NS5_{y1234} - P_y$)

$$\begin{aligned} ds^2_{10} &= H_1\left[dy - \frac{c_1 s_1 H}{H_1}(dt + c_1c_5\mathcal{A}) + s_1c_5\mathcal{A}\right]^2 \\ &\quad -\frac{(1-H)}{H_1}\left[dt + c_1c_5\mathcal{A}\right]^2 + H_5 ds^2_4 + ds^2_{T^4}, \\ B^{(2)} &= c_5s_5\mathcal{C} + s_5\mathcal{B}\wedge(c_1 dt - s_1 dy), \\ e^{2\phi} &= H_5. \end{aligned} \tag{3.41}$$

3.1.8 T-duality along y $(NS5_{y1234} - F1_y)$

$$\begin{aligned}
ds^2_{10} &= H_1^{-1}\left[dy + s_1 s_5 \mathcal{B}\right]^2 - \frac{(1-H)}{H_1}\left[dt + c_1 c_5 \mathcal{A}\right]^2 + H_5 ds_4^2 + ds^2_{T^4}, \\
B^{(2)} &= c_5 s_5 \mathcal{C} + \frac{c_1 s_5 \mathcal{B}}{H_1} \wedge dt + \left[\frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) - s_1 c_5 \mathcal{A}\right] \wedge dy \qquad (3.42)\\
&\quad + s_5 c_5 s_1^2 \frac{1-H}{H_1} \mathcal{B} \wedge \mathcal{A}, \\
e^{2\phi} &= \frac{H_5}{H_1}.
\end{aligned}$$

3.1.9 S-duality $(D5_{y1234} - D1_y)$

$$\begin{aligned}
ds^2_{10} &= H_1^{-1/2} H_5^{-1/2}\left(\left[dy + s_1 s_5 \mathcal{B}\right]^2 - (1-H)\left[dt + c_1 c_5 \mathcal{A}\right]^2\right) \\
&\quad + H_1^{1/2} H_5^{1/2} ds_4^2 + \frac{H_1^{1/2}}{H_5^{1/2}} ds^2_{T^4}, \\
C^{(2)} &= c_5 s_5 \mathcal{C} + \frac{c_1 s_5 \mathcal{B}}{H_1} \wedge dt + \left[\frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) - s_1 c_5 \mathcal{A}\right] \wedge dy \qquad (3.43)\\
&\quad + s_5 c_5 s_1^2 \frac{1-H}{H_1} \mathcal{B} \wedge \mathcal{A}, \\
e^{2\phi} &= \frac{H_1}{H_5}. \qquad (3.44)
\end{aligned}$$

This is the final result: it describes the non-extremal geometry with P_z, KK_z, $D1_y$ and $D5_{y1234}$ charges.

3.2 Change of gauge

It is convenient for later purposes to make a coordinate transformation

$$y' = y - s_1 s_5 \frac{Q}{P} z = y - s_1 s_5 \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}} z. \qquad (3.45)$$

If we combine this with a gauge transformation

$$C^{(2)} \to C^{(2)} - c_1 s_5 \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}} dt \wedge dz, \qquad (3.46)$$

this will leave the metric and two-form gauge field in the same form as before, but with a shifted $\mathcal{B}$:

$$\mathcal{B}' = \mathcal{B} + \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}} dz = -(V_0 - \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}})(dz + \omega^1) + (\kappa_0^1 - \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}} \omega^1). \qquad (3.47)$$

We would like to re-absorb this shift into a redefinition of V_0 and κ_0^1 as indicated; since

$$\mathcal{C} = (dz + \omega^1) \wedge [-V_0 \omega^0 + {\kappa^0}_0], \tag{3.48}$$

This also involves shifting κ_0^0,

$$\kappa_0^{0\prime} = \kappa_0^0 - \sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}}\omega^0. \tag{3.49}$$

Thus, the solution is of the same form as before after this transformation, but with the new quantities

$$V_0 = -\frac{1}{A}\sqrt{\frac{q(q^2+m^2)}{p(p^2+m^2)}}\left[f^2 + 2p\left(\rho + p + \frac{qb}{m}\sqrt{\frac{p^2+m^2}{q^2+m^2}}\cos\theta\right)\right], \tag{3.50}$$

$$\kappa_0^1 = \frac{2b\sqrt{q}\sqrt{p+q}}{m\sqrt{p^2+m^2}}\frac{\sin^2\theta}{f^2}[\rho(pq+m^2) + m^2(p-q)], \tag{3.51}$$

$$\kappa_0^0 = -\frac{2}{f^2}q\left[\Delta\cos\theta + \sqrt{\frac{q^2+m^2}{p^2+m^2}}\frac{b}{m}(p\rho - m^2)\sin^2\theta\right]. \tag{3.52}$$

Henceforth we will always work in this coordinate system, and will omit the prime on y.

3.3 Summary of the solution

We have now constructed an appropriate solution carrying the required charges. Let us collect together some information about the solution here for ease of reference. As in [31], it will be convenient for studying the singularity structure to rewrite the solution by writing factors of A explicitly. Let us therefore write $(1-H) = G/A$, $H_{1,5} = \tilde{H}_{1,5}/A$. Then the charged metric can be written as

$$\begin{aligned} ds_{10}^2 &= (\tilde{H}_1\tilde{H}_5)^{-1/2}\left[A(dy + s_1 s_5 \mathcal{B})^2 - G(dt + c_1 c_5 \mathcal{A})^2\right] \\ &\quad + (\tilde{H}_1\tilde{H}_5)^{1/2}\left[\frac{f^2}{AG}(dz+\omega^1)^2 + \frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2\right] \\ &\quad + \frac{\tilde{H}_1^{1/2}}{\tilde{H}_5^{1/2}} ds_{T^4}^2, \end{aligned} \tag{3.53}$$

and the matter fields are

$$C^{(2)} = c_5 s_5 \mathcal{C} + \frac{c_1 s_5 A\mathcal{B}}{\tilde{H}_1}\wedge dt + \left[\frac{c_1 s_1 (A-G)}{\tilde{H}_1}(dt + c_1 c_5 \mathcal{A}) - s_1 c_5 \mathcal{A}\right]\wedge dy + s_5 c_5 s_1^2 \frac{G}{\tilde{H}_1}\mathcal{B}\wedge\mathcal{A}, \tag{3.54}$$

$$e^{2\phi} = \frac{\tilde{H}_1}{\tilde{H}_5}, \tag{3.55}$$

where

$$\mathcal{A} = \omega^0 - \frac{C}{G}(dz + \omega^1), \tag{3.56}$$

$$\mathcal{B} = -V_0(dz + \omega^1) + \kappa_0^1, \tag{3.57}$$

$$\mathcal{C} = (dz + \omega^1) \wedge (-V_0\omega^0 + \kappa_0^0) = dz \wedge (-V_0\omega^0 + \kappa_0^0), \tag{3.58}$$

$$\tilde{H}_{1,5} = A + (A - G)s_{1,5}^2, \tag{3.59}$$

$$G = A(1 - H) = \frac{Af^2 - C^2}{B}. \tag{3.60}$$

The functions from the vacuum metric are given in equations (2.44–2.50), and we work with the shifted y coordinate, so the quantities from the electromagnetic duality are given in (3.50–3.52). The determinant of the metric is

$$g = -\frac{\tilde{H}_1^3}{\tilde{H}_5}\sin^2\theta. \tag{3.61}$$

Since both the y and z directions have a finite size as $\rho \to \infty$, the solution is asymptotically flat in four dimensions. By rearranging the metric, we can rewrite it in a form which is suitable for Kaluza-Klein reduction,

$$\begin{aligned} ds_{10}^2 \;=\; & (\tilde{H}_1\tilde{H}_5)^{-1/2}\left[A(dy + s_1s_5\mathcal{B})^2 + D(dz + \omega^1 + c_1c_5\frac{C}{D}(dt + c_1c_5\omega^0))^2\right] \\ & +(\tilde{H}_1\tilde{H}_5)^{1/2}\left[-\frac{f^2}{AD}(dt + c_1c_5\omega^0)^2 + \frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2\right] \\ & +\frac{\tilde{H}_1^{1/2}}{\tilde{H}_5^{1/2}}ds_{T^4}^2, \end{aligned} \tag{3.62}$$

where

$$D = Bc_1^2c_5^2 - f^2(c_1^2s_5^2 + s_1^2c_5^2) + \frac{Gf^2}{A}s_1^2s_5^2. \tag{3.63}$$

The charges of the four-dimensional asymptotically flat solution are

$$\begin{aligned} \mathcal{M} &= \frac{1}{2}[p + q(1 + s_1^2 + s_5^2)], \\ \mathcal{P} &= P = \sqrt{\frac{p(p^2 + m^2)}{p + q}}, \\ \mathcal{Q} &= Qc_1c_5 = \sqrt{\frac{q(q^2 + m^2)}{(p + q)}}c_1c_5, \\ \mathcal{J} &= Jc_1c_5 = b\frac{\sqrt{pq}(pq - m^2)}{m(p + q)}c_1c_5, \\ \mathcal{Q}_i &= qs_ic_i, \quad i = 1, 5. \end{aligned} \tag{3.64}$$

Here $\mathcal{M}$ is the mass of the solution and $\mathcal{J}$ its angular momentum, expressed in units for which $G_4 = 1$; $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}_1$ and $\mathcal{Q}_5$ denote the KK monopole, KK electric, D1 and D5 charges.

3.4 BPS limit

Let us consider the limit of the geometry (3.53) in which $m \to 0$, with the charges and the angular momentum held fixed. If the charges $\mathcal{Q}_1$ and $\mathcal{Q}_5$ are fixed to non-zero values, then the boost parameters δ_1 and δ_5 must be taken to infinity. The resulting geometry can be parametrized by its charges, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}_1$ and $\mathcal{Q}_5$ (assumed to be positive), and by the angular momentum parameter b, all of which are finite in this limit. One finds, using Eqs. (3.64), that p, q and δ_i should behave as

$$p = \mathcal{P} + O(m),\ q = \left(\frac{m}{\mathcal{Q}}\right)^2 \frac{\mathcal{Q}_1\mathcal{Q}_5}{\mathcal{P}} + O(m^3),\ \sinh\delta_i = \frac{\mathcal{Q}}{m}\sqrt{\frac{\mathcal{P}\mathcal{Q}_i}{\mathcal{Q}_1\mathcal{Q}_5}} + O(m^0),\ i = 1, 5. \tag{3.65}$$

In this limit the mass of the solution reduces to the sum of the D1, D5 and KK monopole charges:

$$\mathcal{M} = \frac{1}{2}[\mathcal{P} + \mathcal{Q}_1 + \mathcal{Q}_5]. \tag{3.66}$$

This shows that the limit $m \to 0$ saturates the BPS bound.

Let us introduce the new coordinates

$$\tilde{r} = \rho - b\cos\theta\,,\ \cos\tilde{\theta} = \frac{\rho\cos\theta - b}{\rho - b\cos\theta}. \tag{3.67}$$

The metric, gauge field and dilaton one obtains after performing the limit (3.65) can be recast in the form

$$\begin{aligned}
ds^2 &= (Z_1 Z_5)^{-1/2}[-(dt-k)^2 + (dy + \omega_P - k)^2] + (Z_1 Z_5)^{1/2} ds_B^2 + \left(\frac{Z_1}{Z_5}\right)^{1/2} ds^2_{T^4},\\
ds_B^2 &= V^{-1}(dz+\chi)^2 + V(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2\sin^2\tilde{\theta}^2 d\phi^2),\\
C^{(2)} &= \vec{Z}_5 \wedge (dz+\chi) + (dy + dt + \omega_P)\wedge\left(\frac{dt+k}{Z_1}\right)\\
e^{2\Phi} &= \frac{Z_1}{Z_5}.
\end{aligned} \tag{3.68}$$

Here Z_1, Z_5, and V are harmonic functions on the flat three-dimensional space spanned by the coordinates $\tilde{r}$, $\tilde{\theta}$ and ϕ; k and ω_P are 1-forms on the four-dimensional space with metric ds_B^2, of the form

$$k = \left(H_k + \frac{H_P}{2V}\right)(dz+\chi) + \vec{k}\,,\ \omega_P = \frac{H_P}{V}(dz+\chi) + \vec{\omega}_P, \tag{3.69}$$

where H_k and H_P are harmonic functions and $\vec{k}$ and $\vec{\omega}$ are 1-forms on $\mathbb{R}^3$ that satisfy

$$*_3\, d\vec{k} = V dH_k - H_k dV - \frac{dH_P}{2}\,,\ *_3 d\vec{\omega}_P = -dH_P, \tag{3.70}$$

with $*_3$ the Hodge dual on $\mathbb{R}^3$; χ and $\vec{Z}_5$ are 1-forms on $\mathbb{R}^3$ related to V and Z_5 by

$$*_3\, d\chi = dV\ ,\ *_3\, d\vec{Z}_5 = dZ_5. \tag{3.71}$$

Now (3.68) is of the general form of a supersymmetric solution with a Gibbons-Hawking base space, and vanishing momentum along y. This general form was obtained in [30]. This shows that in the $m \to 0$ limit the solution (3.53) becomes supersymmetric.

The explicit values of the functions V, Z_i, H_k and H_P, which are obtained by taking this limit of (3.53) are:

$$\begin{aligned} &V = 1 + \frac{Q_K}{\tilde{r}}\,,\ Z_i = 1 + \frac{Q_i}{\tilde{r}_c}\,,\ i = 1,5 \\ &H_k = -\frac{Q_{Ke}}{2Q_K}\Big(1 + \frac{Q_K}{\tilde{r}_c}\Big)\,,\ H_P = \frac{Q_{Ke}}{Q_K} + \frac{Q_1 Q_5}{Q_{Ke}}\Big(\frac{1}{\tilde{r}} - \frac{1}{\tilde{r}_c}\Big), \end{aligned} \tag{3.72}$$

where we have defined

$$\begin{aligned} &c = 2b\,,\ Q_K = 2\mathcal{P}\,,\ Q_{Ke} = 2\mathcal{Q}\,,\ Q_i = 2\mathcal{Q}_i\,,\ i = 1,5 \\ &\tilde{r}_c = \sqrt{\tilde{r}^2 + c^2 + 2c\tilde{r}\cos\tilde{\theta}}. \end{aligned} \tag{3.73}$$

Let us review the analysis of the singularity structure of the supersymmetric metric (3.68). A general analysis of the regularity of metrics of the form (3.68) has been performed in [25, 18, 19, 26, 27]. One should ensure that the 1-forms k and ω_P are regular at the point $\tilde{r} = 0$, where the KK monopole potential V diverges,. This, in particular, requires that

$$k_z = H_k + \frac{H_P}{2V} = 0 \tag{3.74}$$

at $\tilde{r} = 0$. This condition is satisfied if

$$c = \frac{Q_K Q_{Ke}^2}{Q_1 Q_5 - Q_{Ke}^2}. \tag{3.75}$$

It can be checked that, with the condition (3.75), the metric (3.68, 3.72) is regular if the coordinates y and z are subject to the identifications[1]

$$\begin{aligned} &(y, z) \sim (y + 2\pi R_y, z) \sim (y, z + 2\pi R_z) \\ &R_y = 2\frac{Q_1 Q_5}{Q_{Ke}}\,,\ R_z = 2\frac{Q_K}{N_K}\,,\ N_K \in \mathbb{N}. \end{aligned} \tag{3.76}$$

This metric with these identifications coincides with the smooth supersymmetric D1-D5-KK solution found in [25] by a completely different method, i.e. by adding KK charge to the extremal D1-D5 geometry of [3, 4].

[1]The metric is strictly speaking regular only for $N_K = 1$. For N_K integer greater than one, the metric has the usual conical singularity corresponding to N_K coinciding monopoles.

4 Finding smooth solutions

Within the family of metrics constructed in the previous section, we want to see whether there are any smooth solutions. We can see by inspection that the metric will have coordinate singularities at $\tilde{H}_1 = 0$, $\tilde{H}_5 = 0$, $\theta = 0, \pi$ and $\Delta = 0$. Because $\tilde{H}_{1,5}$ involve $1/B$, it will also have singularities at $B = 0$. Although the form of the metric in (3.53) appears to involve factors of $1/G$, these cancel out in the actual metric coefficients, as can be seen from the alternative form (3.62), so there is no problem at $G = 0$. There is a potential coordinate singularity at $A = 0$. There is also a potential singularity at $f^2 = 0$, but since $f^2 = \Delta + b^2 \sin^2\theta$, we will always meet a singularity at $\Delta = 0$ first.

We will focus on the singularity at $\rho = \rho_0 = \sqrt{b^2 - m^2}$, where $\Delta = 0$, and try to interpret it as a smooth origin. As usual, $\theta = 0, \pi$ should be coordinate singularities. This will require appropriate identifications, to be analysed later. We would expect that the other coordinate singularities would be true curvature singularities, so we wish to arrange to have solutions where $\tilde{H}_1, \tilde{H}_5, A, B > 0$ everywhere. The determinant of the metric on the surfaces of constant ρ vanishes at $\Delta = 0$. For the case with no momentum charge which we are studying in this paper, we require the identifications of y and z to lie in the surfaces of constant t. Hence for $\rho = \rho_0$ to be a smooth origin, we need the determinant of the metric on the surfaces of constant ρ and t to also vanish there. This determinant can be easily evaluated using (3.62):

$$g_{(\rho t)} = \frac{\tilde{H}_1^2}{\tilde{H}_5^2 f^2} \left[A\Delta D \sin^2\theta - c_1^2 c_5^2 (f^2 \omega_\phi^0)^2 \right]. \tag{4.1}$$

In particular, at $\Delta = 0$, this is a non-zero factor times the square of

$$f^2 \omega_\phi^0 = 2J \sin^2\theta \left(\rho - \frac{m^2(p+q)}{(pq - m^2)} \right). \tag{4.2}$$

Therefore, for the determinant to vanish at $\rho = \rho_0$, we need

$$\rho_0 = \sqrt{b^2 - m^2} = \frac{m^2(p+q)}{(pq - m^2)}. \tag{4.3}$$

This implies

$$b^2 = \frac{m^2(p^2 + m^2)(q^2 + m^2)}{(pq - m^2)^2}. \tag{4.4}$$

We will always assume we take the positive square root. If the parameters satisfy (4.4), the singularity at $\Delta = 0$ is a degeneration, where one of the spatial directions is going to zero size.

We should check that no other singularity will be encountered in the region $\rho \geq \rho_0$, $0 \leq \theta \leq \pi$. Using (4.4), we can rewrite

$$A = f^2 + 2p \left[(\rho - \rho_0) + \frac{b^2}{\rho_0}(1 + \cos\theta) \right], \tag{4.5}$$

$$B = f^2 + 2q\left[(\rho-\rho_0) + \frac{b^2}{\rho_0}(1-\cos\theta)\right], \tag{4.6}$$

so we can see that $A > 0$ and $B > 0$ for $\rho > \rho_0$. Also,

$$\tilde{H}_i = Ac_i^2 - Gs_i^2 = A\left(c_i^2 - \frac{f^2}{B}s_i^2\right) + \frac{C^2}{B}s_i^2 > 0 \tag{4.7}$$

for $\rho > \rho_0$, as $A > 0$ and $B > f^2$. Thus, when (4.4) is satisfied, the only singularities in the metric are at $\rho = \rho_0$ and at $\theta = 0, \pi$. Each of these is a degeneration in the (y, z, ϕ) part of the metric.

4.1 Identifications

So far, we have performed a local analysis. We now want to see what global identifications we need to make in the (y, z, ϕ) space to have a smooth metric. At each of the three coordinate singularities, $\rho = \rho_0$, $\theta = 0$, or $\theta = \pi$, some combination of these directions is going to zero size, and we want to choose an appropriate period to make this a smooth origin in a plane (we could in general allow orbifold singularities, but for simplicity we focus on the task of constructing smooth metrics). We will write a general Killing vector in this space as $\xi = \partial_\phi - \alpha\partial_y - \beta\partial_z$, and choose α and β to make the norm of the Killing vector vanish at the degeneration in each case. The direction which goes to zero size is then along ϕ at fixed $y + \alpha\phi$, $z + \beta\phi$. In each case, it will turn out that we have to set $\alpha = s_1 s_5 \kappa^1_{0,\phi}$, $\beta = \omega^1_\phi$ to make the contributions to $\xi\cdot\xi$ from the first line in (3.62) vanish.

Consider first the singularities at $\theta = 0, \pi$. At $\theta = 0$, $f^2 = \Delta$, $\omega^0 = 0$, $\kappa^1_0 = 0$, and

$$\omega^1 = 2\frac{\sqrt{p}\sqrt{p^2+m^2}}{\sqrt{p+q}}d\phi = 2\mathcal{P}d\phi. \tag{4.8}$$

Thus the direction which goes to zero size at $\theta = 0$ is along ϕ at fixed $z + 2\mathcal{P}\phi$, y. The metric looks locally like $d\theta^2 + \sin^2\theta d\phi^2$, so ϕ needs to be a 2π periodic coordinate. Thus, the identification required to make this a smooth origin is[2]

$$(y, z, \phi) \sim (y, z - 4\pi\mathcal{P}, \phi + 2\pi). \tag{4.9}$$

Similarly, at $\theta = \pi$, $f^2 = \Delta$, $\omega^0 = 0$, $\kappa^1_0 = 0$, and $\omega^1 = -2\mathcal{P}d\phi$, so the direction which goes to zero size at $\theta = \pi$ is along ϕ at fixed $z - 2\mathcal{P}\phi$, y, and the required identification is

$$(y, z, \phi) \sim (y, z + 4\pi\mathcal{P}, \phi + 2\pi). \tag{4.10}$$

Finally, at $\rho = \rho_0$, $f^2 = b^2\sin^2\theta$, $\omega^0 = 0$, $\omega^1 = -2\mathcal{P}d\phi$,

$$\kappa^1_{0,\phi} = 4q\frac{\sqrt{q}\sqrt{p+q}}{\sqrt{q^2+m^2}}, \tag{4.11}$$

[2]The shift of y by z we introduced in section 3.2 was chosen to make this and the next identification be at constant y.

so the relevant circle is along ϕ at fixed $z - 2\mathcal{P}\phi$, $y + 4s_1 s_5 q \frac{\sqrt{q}\sqrt{p+q}}{\sqrt{q^2+m^2}}$. The leading contribution to the non-zero size of this circle away from $\rho = \rho_0$ comes just from the $\frac{\Delta}{f^2}\sin^2\theta d\phi^2$ term in the metric; the first line of (3.62) makes a contribution of order $(\rho - \rho_0)^2$. Therefore, writing $\rho = \rho_0(1 + 2z^2)$, the relevant part of the metric is

$$4\sqrt{\tilde{H}_1 \tilde{H}_5}(dz^2 + \frac{\rho_0^2}{b^2} d\phi^2). \tag{4.12}$$

Thus, the necessary identification here is

$$(y, z, \phi) \sim (y - 8\pi n s_1 s_5 q \frac{\sqrt{q}\sqrt{p+q}}{\sqrt{q^2+m^2}}, z + 4\pi n \mathcal{P}, \phi + 2\pi n), \tag{4.13}$$

where $n = b/\rho_0$. We will write

$$R_y = 4q \frac{\sqrt{q}\sqrt{p+q}}{\sqrt{q^2+m^2}} s_1 s_5 \tag{4.14}$$

in subsequent expressions for compactness. We want the metric that we obtain by Kaluza-Klein reduction from (3.62) to be asymptotically flat in four dimensions, so after the dimensional reduction, ϕ must be 2π periodic. Given the identification (4.13), this imposes a second condition on the parameters:

$$n = \frac{b}{\sqrt{b^2 - m^2}} \in \mathbb{Z}. \tag{4.15}$$

If we consider a solution satisfying (4.4) and (4.15), and the periodicities (4.9, 4.10, 4.13), the metric will be smooth at the coordinate singularities. We will verify in the next section that it is also smooth in the corners where two circles are going to zero size simultaneously, and that the matter fields are smooth.

It is important to note that the periodicities (4.9, 4.10, 4.13) do not fix the lattice of identifications in the y, z, ϕ space uniquely. This is because although we need each of these identifications to be a primitive vector in the lattice,[3] (4.9, 4.10, 4.13) do not necessarily form a basis for the lattice. Specifying the most general lattice consistent with the requirement that (4.9, 4.10, 4.13) are primitive lattice vectors is quite complicated, so we will not discuss it in detail. As a particular example, this freedom includes the freedom to choose the integer-quantized magnetic Kaluza-Klein charge. We get a solution with N_K units of magnetic Kaluza-Klein charge on reduction to four dimensions by taking the basis of identifications to be

$$(y, x^5, \phi) \sim (y - 2\pi n R_y, z, \phi) \sim (y, z + 8\pi \frac{\mathcal{P}}{N_K}, \phi) \sim (y, z + 4\pi\mathcal{P}, \phi + 2\pi). \tag{4.16}$$

[3]This is necessary to make the metric smooth at the corresponding coordinate singularity. If the identification is not a primitive lattice vector, we will have an orbifold singularity where this cycle degenerates.

This is one example of a large space of possibilities consistent with (4.9, 4.10, 4.13). In the rest of this paper, we will generally proceed as if (4.9, 4.10, 4.13) is a basis of identifications; any other possibility corresponds to taking an orbifold of the spacetime we describe. In particular, more general possibilities may have orbifold singularities in the corners in the ten-dimensional metric.

These smooth solutions admit a unique spin structure, which has antiperiodic boundary conditions for the fermions around each of the contractible cycles (4.9), (4.10) and (4.13). The fermions will thus be periodic under $z \sim z + 8\pi\mathcal{P}$, and will be periodic under $y \sim y - 2\pi n R_y$ for odd n, and antiperiodic for even n. Thus, for odd n, the solutions have a spin structure compatible with preserving supersymmetry at large distances.

4.2 Solving the constraints

There are two constraints on the parameters to obtain a smooth solution, (4.4) and (4.15). It is useful to have an explicit solution of these constraints. We can obtain a simple solution by treating p and ρ_0 as the independent parameters, and solving for everything else in terms of them. We then have

$$b = n\rho_0, \quad m^2 = \rho_0^2(n^2-1), \quad q = \frac{\rho_0(p+\rho_0)(n^2-1)}{(p-\rho_0(n^2-1))}. \tag{4.17}$$

We assume ρ_0, p are such that $q > 0$. The various functions appearing in the solution can be rewritten in terms of these parameters, which makes their positivity properties more manifest:

$$\Delta = \rho^2 - \rho_0^2, \quad f^2 = (\rho^2 - \rho_0^2) + \rho_0^2 n^2 \sin^2\theta, \tag{4.18}$$

$$A = f^2 + 2p[(\rho-\rho_0) + n^2\rho_0(1+\cos\theta)], \tag{4.19}$$

$$B = f^2 + 2\frac{\rho_0(p+\rho_0)(n^2-1)}{(p-\rho_0(n^2-1))}[(\rho-\rho_0) + n^2\rho_0(1-\cos\theta)], \tag{4.20}$$

$$C = \frac{2\rho_0\sqrt{\rho_0(\rho_0+p)}\,n(n^2-1)}{(p-\rho_0(n^2-1))}[(\rho-\rho_0) + (\rho_0+p)(1-\cos\theta)], \tag{4.21}$$

$$\omega^0 = \frac{2J\sin^2\theta(\rho-\rho_0)}{f^2}d\phi, \quad J^2 = \frac{\rho_0^3 p(\rho_0+p)n^2(n^2-1)^2}{(p-\rho_0(n^2-1))}, \tag{4.22}$$

$$\omega^1 = \frac{2}{f^2}\sqrt{p(p-\rho_0(n^2-1))}[(\rho^2-\rho_0^2)\cos\theta - \frac{\rho_0 p n^2}{(p-\rho_0(n^2-1))}(\rho-\rho_0)\sin^2\theta - n^2\rho_0^2\sin^2\theta]d\phi, \tag{4.23}$$

and

$$V_0 = -\frac{n(n^2-1)}{A}\sqrt{\frac{\rho_0^3(p+\rho_0)}{p(p-\rho_0(n^2-1))^3}}[f^2 + 2p(\rho + p + (p+\rho_0)\cos\theta)], \tag{4.24}$$

$$\kappa_0^1 = \frac{2n\sqrt{\rho_0(p+\rho_0)}}{(p-\rho_0(n^2-1))}\frac{\sin^2\theta}{f^2}\Big[\frac{\rho_0(n^2-1)}{(p-\rho_0(n^2-1))}(p^2+2p\rho_0-\rho_0^2(n^2-1))(\rho-\rho_0) \\ +2\rho_0^2(n^2-1)(\rho_0+p)\Big], \tag{4.25}$$

$$\kappa_0^0 = -\frac{2}{f^2}\frac{\rho_0(p+\rho_0)(n^2-1)}{(p-\rho_0(n^2-1))}\left[(\rho^2-\rho_0^2)\cos\theta + \frac{n^2\rho_0}{(p-\rho_0(n^2-1))}(p\rho-\rho_0^2(n^2-1))\sin^2\theta\right]. \tag{4.26}$$

This parametrization will be used later in relating the $n = 1$ case to the supersymmetric solution and to study the near-core decoupling limit of the solutions.

5 Verifying Regularity

5.1 Matter fields

The dilaton is clearly regular. For the gauge field $C^{(2)}$, we would like to see that it is possible to make gauge transformations to make the field regular at each of the degenerations. Recall from section 3.3 that

$$C^{(2)} = c_5 s_5 \mathcal{C} + \frac{c_1 s_5 \mathcal{B}}{H_1}\wedge dt - \left[-\frac{c_1 s_1 H}{H_1}(dt + c_1 c_5 \mathcal{A}) + s_1 c_5 \mathcal{A}\right]\wedge dy + s_5 c_5 s_1^2 \frac{1-H}{H_1}\mathcal{B}\wedge\mathcal{A}, \tag{5.1}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are given in (3.56, 3.57, 3.58). We need to calculate the component of this two-form along the degenerating direction $\xi = \partial_\phi - \alpha\partial_y - \beta\partial z$. This is

$$\begin{aligned} \mathrm{i}_\xi C^{(2)} = &\left[c_1 s_5(-V_0\omega_\phi^1 + \kappa_{0,\phi}^1) + \alpha c_1 s_1 H + \beta c_1 s_5 V_0\right]\frac{dt}{H_1} \\ &- \left[s_1 c_5(\omega_\phi^0 - \frac{C}{G}\omega_\phi^1) + \beta s_1 c_5 \frac{C}{G}\right]\frac{(1-H)}{H_1}dy \\ &+ \left[c_5 s_5(V_0\omega_\phi^0 c_1^2 - \kappa_{0,\phi}^0 H_1 - (1-H)\frac{C}{G}\kappa_{0,\phi}^1 s_1^2) + \alpha s_1 c_5(1-H)\frac{C}{G}\right]\frac{dz}{H_1} \\ &+ \left[-\alpha s_1 c_5(1-H)(\omega^0 - \frac{C}{G}\omega^1) - \beta c_5 s_5(-V_0\omega^0 c_1^2 + \kappa_0^0 H_1 + \frac{C}{G}(1-H)\kappa_0^1 s_1^2)\right]\frac{1}{H_1}. \end{aligned} \tag{5.2}$$

Since at each degeneration, $\alpha = s_1 s_5 \kappa_{0,\phi}^1$ and $\beta = \omega_\phi^1$, we can consider the three different degenerations simultaneously by substituting in these values of α and β. Substituting these in,

$$\begin{aligned} \mathrm{i}_\xi C^{(2)} = &c_1 s_5 \kappa_{0,\phi}^1 dt - s_1 c_5 \frac{1-H}{H_1}\omega_\phi^0 dy + \left(\frac{c_5 s_5 c_1^2 V_0 \omega_\phi^0 c_1^2}{H_1} - c_5 s_5 \kappa_{0,\phi}^0\right)dz \\ &+ \left(\frac{s_5 c_5(s_1^2(H-1)\kappa_{0,\phi}^1 + c_1^2 V_t \omega_\phi^1)\omega_\phi^0}{H_1} - c_5 s_5 \omega_\phi^1 \kappa_{0,\phi}^0\right)d\phi. \end{aligned} \tag{5.3}$$

Note that this expression is valid only near one of the coordinate singularities. Since $\omega^0 = 0$ at each of these, this expression simplifies to

$$\mathrm{i}_\xi C^{(2)} = c_1 s_5 \kappa^1_{0,\phi} dt - c_5 s_5 \kappa^0_{0,\phi}(dz + \omega^1_\phi d\phi). \tag{5.4}$$

These remaining terms are all constants. Thus, these components of $C^{(2)}$ are locally pure gauge, and it looks like we ought to be able to remove them by a gauge transformation to obtain a two-form potential which is regular at the degeneration.

However, this may not be possible globally. The integral of $C^{(2)}$ over a closed two-cycle is gauge-invariant, and if there is a non-zero integral over a two-cycle which shrinks to zero size, it will indicate a singularity in the gauge field. We therefore need to consider whether there is any such integral which is non-zero. Since the component of $C^{(2)}$ along the degenerating direction never has a non-zero dy component, the integrals to consider are where we integrate over the degenerating cycle and one of the two cycles (4.9), (4.10). Here we need to consider the cases separately. If the cycle (4.9) is degenerating, then $\omega^1 = 2\mathcal{P} d\phi$, and the integral over the 2-cycle formed by the product of the 1-cycles (4.9) and (4.10) is

$$\oint C^{(2)} = -16\pi^2 c_5 s_5 \mathcal{P} \kappa^0_{0,\phi} = 32\pi^2 \mathcal{P} \mathcal{Q}_5. \tag{5.5}$$

When it is the cycle (4.10) which is degenerating, $\omega^1 = -2\mathcal{P} d\phi$, and the integral over the 2-cycle determined by the cycles (4.9) and (4.10) has the same value as in Eq. (5.5). When it is (4.13) which is degenerating, $\omega^1 = -2\mathcal{P} d\phi$, so the integral over the product of (4.13) and (4.10) vanishes, while the integral over (4.13) and (4.9) has the same value as in (5.5).

These non-zero integrals of $C^{(2)}$ do not immediately imply a singularity in the gauge field, as there is still the freedom to make large gauge transformations. That is, the gauge potential (and hence the integral) actually take values in a circle rather than the reals. If the right-hand side of (5.5) is an integer multiple of the size of the gauge group, it can be set to zero by a large gauge transformation.

The requirement that (5.5) is an integer multiple of the size of the gauge group is in fact just the usual quantization of a magnetic charge, required to make the gauge field well-defined over the whole sphere at large distance. Let us review the usual form of this argument. The magnetic charge associated with $C^{(2)}$ is the integral of the three-form field strength over the surface spanned by (θ, z, ϕ). If we work in a fixed gauge, we can write this integral as

$$\oint_{\theta z \phi} F^{(3)} = \oint_{z\phi} C^{(2)}|_{\theta=\pi} - \oint_{z\phi} C^{(2)}|_{\theta=0}. \tag{5.6}$$

At $\theta = 0, \pi$, the $dz \wedge d\phi$ component of $C^{(2)}$ from (3.54) is simply $C^{(2)}_{z\phi}|_{\theta=0,\pi} = c_s s_5 dz \wedge \kappa^0_0 = \mp 2 q c_5 s_5 dz \wedge d\phi = \mp 2\mathcal{Q}_5 dz \wedge d\phi$. Thus, integrating over (4.9) and (4.10),

$$\oint_{\theta z \phi} F^{(3)} = 32\pi^2 \mathcal{P} \mathcal{Q}_5. \tag{5.7}$$

Now in this gauge, the gauge field is not well-behaved at either end of the range. If we change the gauge so $C^{(2)}_{z\phi}|_{\theta=0} = 0$, then since the charge is gauge-invariant, we will have $C^{(2)}_{z\phi}|_{\theta=\pi} = 4\mathcal{Q}_5 dz \wedge d\phi$. For the gauge field to be globally well-behaved on the whole surface, there must be a large gauge transformation which can be used to shift this to zero. This is equivalent to requiring that the charge (5.7) is a multiple of the size of the gauge group. This large gauge transformation is then precisely what we need to see that the integral (5.5) of the two-form over the degenerating two-cycles is gauge-equivalent to zero. Thus, we have succeeded in showing that the gauge field is regular up to gauge transformations.

5.2 Corners

With the conditions above, the solution is smooth at $\rho = \rho_0$ or $\theta = 0, \pi$. However, it is not clear what happens in the 'corners', where $\rho = \rho_0$ and $\theta = 0, \pi$. In this section, we will introduce coordinates which explicitly show that the ten-dimensional geometry is smooth at these points as well.

Consider first the corner at $\rho = \rho_0$, $\theta = 0$. Define new coordinates by[4]

$$\tilde{r} = (\rho - \rho_0) + \rho_0(1 - \cos\theta), \tag{5.8}$$

$$\tilde{r}\cos^2\frac{\tilde{\theta}}{2} = \frac{\rho - \rho_0}{2}(1 + \cos\theta). \tag{5.9}$$

In the new coordinates, $\rho = \rho_0$, $\theta = 0$ is at $\tilde{r} = 0$, with $\rho = \rho_0$, $\theta \neq 0$ along $\tilde{\theta} = \pi$, and $\rho \neq \rho_0$, $\theta = 0$ along $\tilde{\theta} = 0$. In these coordinates,

$$\frac{d\rho^2}{\Delta} + d\theta^2 = \frac{1}{\tilde{r}\tilde{r}_c}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2), \tag{5.10}$$

with $\tilde{r}_c^2 = \tilde{r}^2 + 4\tilde{r}\rho_0\cos\tilde{\theta} + 4\rho_0^2$. Near $\tilde{r} = 0$,

$$\rho - \rho_0 \approx \frac{\tilde{r}}{2}(1 + \cos\tilde{\theta}), \quad \sin^2\theta \approx \frac{\tilde{r}}{\rho_0}(1 - \cos\tilde{\theta}), \tag{5.11}$$

so

$$\Delta \approx \tilde{r}\rho_0(1 + \cos\tilde{\theta}), \tag{5.12}$$

$$f^2 \approx 2\tilde{r}\rho_0\gamma, \tag{5.13}$$

where

$$\gamma = \frac{1}{2}[(1 + \cos\tilde{\theta}) + n^2(1 - \cos\tilde{\theta})]. \tag{5.14}$$

We also have

$$A \approx \frac{4pb^2}{\rho_0}, \tag{5.15}$$

[4] Note that for $n = 1$, these are the same as the coordinates used in section 3.4.

$$B \approx 2\tilde{r}(\rho_0 + q)\gamma, \tag{5.16}$$

$$C \approx \tilde{r}\frac{\sqrt{q}\sqrt{q^2+m^2}}{\sqrt{p+q}}\left[(1+\cos\tilde{\theta}) + \frac{\rho_0+p}{\rho_0}(1-\cos\tilde{\theta})\right]. \tag{5.17}$$

The above scalings imply that G, and hence $\tilde{H}_{1,5}$, remain finite as $\tilde{r} \to 0$: the vanishing of B in the denominator of (3.60) is cancelled by the factor of A, C^2 in the numerator. Also, $\tilde{H}_{1,5}$ are constants, as

$$G \approx \frac{Af^2}{B} \approx \frac{A\rho_0}{\rho_0+q}. \tag{5.18}$$

The one-form $\mathcal{A} \sim \mathcal{O}(\tilde{r})$, so we can ignore it, while

$$\mathcal{B} \approx 2q\sqrt{\frac{q(p+q)}{q^2+m^2}}\left(d\phi + \frac{dz}{2\mathcal{P}}\right). \tag{5.19}$$

Hence the first line in (3.53) just involves constants in this limit. After some algebra, the non-constant part of the metric becomes

$$\frac{f^2}{AG}(dz+\omega^1)^2 + \frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2 \approx \frac{1}{2\rho_0\tilde{r}}\Bigg[d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 \tag{5.20}$$
$$+ \frac{\tilde{r}^2}{2n^2}(1+\cos\tilde{\theta})\left(d\phi + \frac{dz}{2\mathcal{P}}\right)^2 + \frac{\tilde{r}^2}{2}(1-\cos\tilde{\theta})\left(d\phi - \frac{dz}{2\mathcal{P}}\right)^2\Bigg].$$

Thus, if we define coordinates $\tilde{r} = R^2$, $\tilde{\theta}_c = 2\vartheta$,

$$\psi_1 = \frac{1}{2}\left(\phi - \frac{z}{2\mathcal{P}}\right), \quad \psi_2 = \frac{1}{2n}\left(\phi + \frac{z}{2\mathcal{P}}\right), \tag{5.21}$$

$$\hat{y} = y + \frac{R_y}{2}\left(\phi + \frac{z}{2\mathcal{P}}\right), \tag{5.22}$$

the non-constant part of the metric becomes the standard metric on $\mathbb{R}^4$, while the identifications (4.9, 4.10, 4.13) become respectively $\psi_1 \sim \psi_1 + 2\pi$, $\hat{y} \sim \hat{y} + 2\pi R_y$, $\psi_2 \sim \psi_2 + 2\pi$. Thus, the local geometry is globally $\mathbb{R}^4$, and hence smooth near this corner.

Consider next the corner at $\rho = \rho_0$, $\theta = \pi$. We similarly define new coordinates

$$\tilde{r}_c = (\rho - \rho_0) + \rho_0(1+\cos\theta), \tag{5.23}$$

$$\tilde{r}_c \cos^2\frac{\tilde{\theta}_c}{2} = \frac{\rho-\rho_0}{2}(1-\cos\theta). \tag{5.24}$$

In the new coordinates, $\rho = \rho_0$, $\theta = \pi$ is at $\tilde{r}_c = 0$, with $\rho = \rho_0$, $\theta \neq \pi$ along $\tilde{\theta}_c = \pi$, and $\rho \neq \rho_0$, $\theta = \pi$ along $\tilde{\theta}_c = 0$. In these coordinates,

$$\frac{d\rho^2}{\Delta} + d\theta^2 = \frac{1}{\tilde{r}\tilde{r}_c}(d\tilde{r}_c^2 + \tilde{r}_c^2 d\tilde{\theta}_c^2), \tag{5.25}$$

where now $\tilde{r}^2 = \tilde{r}_c^2 + 4\rho_0 \tilde{r}_c \cos\tilde{\theta}_c + 4\rho_0^2$. Near $\tilde{r}_c = 0$,

$$\rho - \rho_0 \approx \frac{\tilde{r}_c}{2}(1 + \cos\tilde{\theta}_c), \quad \sin^2\theta \approx \frac{\tilde{r}_c}{\rho_0}(1 - \cos\tilde{\theta}_c). \tag{5.26}$$

so

$$\Delta \approx \tilde{r}_c \rho_0 (1 + \cos\tilde{\theta}_c), \tag{5.27}$$

$$f^2 \approx 2\tilde{r}_c \rho_0 \gamma_c, \tag{5.28}$$

where as before we will define

$$\gamma_c = \frac{1}{2}[(1 + \cos\tilde{\theta}_c) + n^2(1 - \cos\tilde{\theta}_c)]. \tag{5.29}$$

We also have

$$A \approx 2\tilde{r}_c(\rho_0 + p)\gamma_c, \tag{5.30}$$

$$B \approx \frac{4qb^2}{\rho_0}, \tag{5.31}$$

$$C \approx \frac{4q^{3/2}\sqrt{q^2+m^2}(p^2+m^2)}{\sqrt{p+q}(pq-m^2)}. \tag{5.32}$$

Thus for small $\tilde{r}_c$, $G \approx -C^2/B$ is a constant, and $\tilde{H}_{1,5}$ are then constants:

$$\tilde{H}_{1,5} \approx \frac{C^2}{B} s_{1,5}^2 \approx \frac{4q^2(p^2+m^2)}{(pq-m^2)} s_{1,5}^2. \tag{5.33}$$

Also,

$$\frac{f^2}{AG} \approx -\frac{\rho_0}{\rho_0 + p}, \quad \omega^1 \approx -2\mathcal{P} d\phi, \tag{5.34}$$

so the $\frac{f^2}{AG}(dz + \omega^1)^2$ term in the metric is a constant size circle, and the non-constant part of the metric is, up to an overall factor,

$$d\Sigma^2 = \frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2 + \frac{A}{\tilde{H}_1 \tilde{H}_5}(dy - s_1 s_5 V_0 (dz + \omega^1) + s_1 s_5 \kappa_0^1)^2. \tag{5.35}$$

After considerable algebra, this becomes

$$\begin{aligned} d\Sigma^2 \;\approx\;& \frac{1}{2\rho_0 \tilde{r}_c}(d\tilde{r}_c^2 + \tilde{r}_c^2 d\tilde{\theta}_c^2) \\ &+ \frac{\tilde{r}_c}{\rho_0}(1 + \cos\tilde{\theta}_c)\frac{1}{n^2 R_y^2}\left(dy - s_1 s_5 V_0 (dz + \omega^1)\right)^2 \\ &+ \frac{\tilde{r}_c}{\rho_0}(1 - \cos\tilde{\theta}_c)\left(d\phi + \frac{1}{R_y}(dy - s_1 s_5 V_0(dz + \omega^1))\right)^2. \end{aligned} \tag{5.36}$$

If we set $\tilde{r}_c = R^2$, $\tilde{\theta}_c = 2\vartheta$, $\tilde{z} = z - 2\mathcal{P}\phi$,

$$\psi_1 = \left(\phi + \frac{1}{R_y} y\right), \quad \psi_2 = -\frac{1}{nR_y} y, \tag{5.37}$$

this becomes the standard metric on $\mathbb{R}^4$, plus some terms involving $\tilde{z}$ which are small compared to the $\frac{f^2}{AG} d\tilde{z}^2$ factor. The identifications (4.10), (4.13) become in these coordinates simply $\psi_1 \sim \psi_1 + 2\pi$, $\psi_2 \sim \psi_2 + 2\pi$, so the metric is globally $\mathbb{R}^4$, and hence smooth near this corner.

5.3 Closed timelike curves

Finally, we verify the absence of closed timelike curves in this metric. We will do this by showing that t is a global time function, which requires $g^{tt} < 0$ everywhere. A basis of orthonormal vector fields for (3.62) is

$$e^1 = \frac{(\tilde{H}_1 \tilde{H}_5)^{1/4}}{\sqrt{A}} \partial_y, \quad e^2 = \frac{(\tilde{H}_1 \tilde{H}_5)^{1/4}}{\sqrt{D}} (\partial_z + s_1 s_5 V_0 \partial_y), \tag{5.38}$$

$$e^3 = \frac{\sqrt{\Delta}}{(\tilde{H}_1 \tilde{H}_5)^{1/4}} \partial_r, \quad e^4 = \frac{1}{(\tilde{H}_1 \tilde{H}_5)^{1/4}} \partial_\theta, \tag{5.39}$$

$$e^0 = \frac{\sqrt{AD}}{f(\tilde{H}_1 \tilde{H}_5)^{1/4}} (\partial_t - c_1 c_5 \partial_z), \tag{5.40}$$

$$e^5 = \frac{f}{\sqrt{\Delta} \sin\theta (\tilde{H}_1 \tilde{H}_5)^{1/4}} (\partial_\phi - s_1 s_5 \kappa^1_{0,\phi} \partial_y - \omega^1_\phi \partial_z - c_1 c_5 \omega^0_\phi \partial_t), \tag{5.41}$$

plus four more for the T^4. From this, we can compute

$$g^{tt} = -\frac{AD}{f^2 \sqrt{\tilde{H}_1 \tilde{H}_5}} + \frac{f^2}{\Delta \sin^2\theta \sqrt{\tilde{H}_1 \tilde{H}_5}} c_1^2 c_5^2 (\omega^0_\phi)^2. \tag{5.42}$$

To show this is negative, we will write it in terms of separate factors independent of the charges, and show that each of the factors is negative separately. Let us write $g^{tt} = -\frac{1}{f^2 \Delta \sqrt{\tilde{H}_1 \tilde{H}_5}} U$, where

$$U = F_1 + (s_1^2 + s_5^2) F_2 + c_1^2 c_5^2 F_3 \tag{5.43}$$

and

$$F_1 = (1 + H) A f^2 \Delta, \quad F_2 = H A f^2 \Delta, \tag{5.44}$$

$$F_3 = A(B - f^2)\Delta - H A f^2 \Delta - 4J^2 \sin^2\theta (\rho - \rho_0)^2. \tag{5.45}$$

We already know that A, B, f^2 and Δ are positive for $\rho > \rho_0$. It is also easy to see that H is positive, as

$$H = \frac{(B - f^2)}{B} + \frac{C^2}{AB} > 0 \tag{5.46}$$

because $B > f^2$. This implies that $F_1, F_2 > 0$. It remains to be shown that $F_3 > 0$.

To show that this last term is also positive, we rewrite it as

$$F_3 = \frac{S}{B} = \frac{1}{B}\left[(A(B - f^2)^2 - C^2 f^2)\Delta - BJ^2(\rho - \rho_0)^2 \sin^2\theta\right]. \tag{5.47}$$

As we already know that $B > 0$, it is sufficient to show that the term S is positive. This term is a sixth order polynomial in $r = (\rho - \rho_0)$,

$$S = c_6 r^6 + c_5 r^5 + c_4 r^4 + c_3 r^3 + c_2 r^2 + c_1 r + c_0. \tag{5.48}$$

To prove that $S \geq 0$ it is sufficient (though not necessary) to show that the individual coefficients c_i are positive. We find

$$c_6 = \frac{4q(pq - m^2)}{p+q}, \tag{5.49}$$

$$c_5 = 8q(pq + 2m^2), \tag{5.50}$$

$$c_4 = c_{40} + c_{41}\cos\theta + c_{42}\cos^2\theta, \tag{5.51}$$

$$c_3 = c_{30} + c_{31}\cos\theta + c_{32}\cos^2\theta, \tag{5.52}$$

$$c_2 = \frac{4m^2(1-\cos\theta)q(p^2+m^2)^2(q^2+m^2)}{(p+q)^2(pq-m^2)^3}\left[c_{20} + c_{21}\cos\theta + c_{22}\cos^2\theta + c_{23}\cos^3\theta\right], \tag{5.53}$$

$$c_1 = \frac{8m^2q^2\sin^2\theta(1-\cos\theta)(p^2+m^2)^3(q^2+m^2)^2[m^2(p+q)(1+\cos\theta) + 2q(pq-m^2)]}{(p+q)^2(pq-m^2)^4}, \tag{5.54}$$

$$c_0 = 0. \tag{5.55}$$

In the following we will not need the explicit values of the coefficients c_{2i}, c_{3i} and c_{4i}. Noting that $p > 0, q > 0$ and $pq - m^2 > 0$, the first two coefficients are immediately seen to be positive. For c_4, c_3 and c_2 the story is more complicated, and it turned out to be simplest to show these terms are positive indirectly. Let $x \equiv \cos\theta$. Then for c_4 we have

$$\begin{aligned} c_4(-1) &= (p+q)^{-2}(pq-m^2)^{-1}\left[16pq((p^2+3qp+4q^2)m^4\right. \\ &\quad \left.+q(2p^3+5qp^2+3q^2p+3q^3)m^2 + p^2q^3(p+2q)) + 16m^4q^2\left(pq-m^2\right)\right] > 0, \end{aligned} \tag{5.56}$$

$$\begin{aligned} c_4(1) &= \frac{16q}{(p+q)(pq-m^2)}\left[\left(p^2+3qp+q^2\right)m^4 + 2pq\left(p^2+qp+q^2\right)m^2\right. \\ &\quad \left.+p^3q^3\right] > 0, \end{aligned} \tag{5.57}$$

$$c_4''(x) = -\frac{16m^2q(p^2+m^2)(q^2+m^2)}{(p+q)(pq-m^2)} < 0. \tag{5.58}$$

From this data one can see that c_4 is positive at the boundaries, and is an inverted parabola. This implies that $c_4(x) > 0$ for all x between -1 and 1. Similar data for c_3 also proves that it is positive for all values of θ:

$$\begin{aligned} c_3(-1) &= \frac{32q^2(p^2+m^2)(q^2+m^2)}{(p+q)^2(pq-m^2)^2}\left[p^3q^2+m^2p(p^2+pq+q^2)\right. \\ &\quad \left.+m^2(p+2q)(pq-m^2)\right] > 0, \quad (5.59) \\ c_3(1) &= \frac{32m^2pq^2(p^2+m^2)(q^2+m^2)}{(pq-m^2)^2} > 0, \quad (5.60) \\ c_3''(x) &= -\frac{16m^2q(p^2+m^2)(q^2+m^2)(pq(2q+p)+m^2(2p+3q))}{(p+q)(pq-m^2)^2} < 0. \quad (5.61) \end{aligned}$$

For c_2 the argument is more complicated. The prefactors in (5.53) are clearly positive so it is only necessary to consider the bracketed term. Let us call this term $\tilde{c}_2$. We first prove that $\tilde{c}_2'(x)$ is positive for $x \in [-1,1]$. This we do by furnishing the same data as done above for c_4 and c_3:

$$\begin{aligned} \tilde{c}_2'(-1) &= 4q(pq^2(p+2q)+pm^2(2p+3q)+m^2(pq-m^2)) > 0, \quad (5.62) \\ \tilde{c}_2'(1) &= 4q\left[\left(p^2+pq+q^2\right)m^2+p^2q^2\right]+4m^2p(pq-m^2) > 0, \quad (5.63) \\ \tilde{c}_2''' &= -6m^2(p+q)(q^2+m^2) < 0. \quad (5.64) \end{aligned}$$

Now given that $\tilde{c}_2'(x)$ is positive for $x \in [-1,1]$ it is sufficient to show that $\tilde{c}_2(-1) > 0$ in order to prove that $\tilde{c}_2$ is positive for all θ. We find

$$\tilde{c}_2(-1) = \frac{64m^2q^3(p^2+m^2)^2(q^2+m^2)}{(p+q)(pq-m^2)^2} > 0. \quad (5.65)$$

Finally for c_1 it is clear from (5.54) that it is positive for all θ. Thus we have shown that S is positive for all values of $\rho > \rho_0$ and θ. Hence $g^{tt} < 0$, so t is a global time function, and there are no closed timelike curves in the geometry.

Note also from (5.42) that $g^{tt} < 0$ implies $D > 0$. This will be significant in the analysis of the four-dimensional solution.

6 Properties of the solutions

6.1 The supersymmetric case

In the special case $n = 1$, we would expect to recover the supersymmetric solution of [25]. From the parametrization (4.17), we can see that $m \to 0$ with q/m^2 and b fixed as $n \to 1$. Thus, if we scale $\delta_i \to \infty$ so as to keep $m^2 s_i c_i$ fixed as we take $n \to 1$, we will be taking the extremal limit described in (3.65). In this limit the constraint (4.15) reduces to the regularity condition (3.75) we have found in Section 3.4 for the supersymmetric solution.

Also the identifications (4.16) become equivalent to (3.76). Thus one can think of the regular supersymmetric geometry (3.68, 3.72) as the particular member of the class of smooth metrics of Section 4 with $n = 1$, provided that one also takes the δ_i parameters to infinity, as specified in (3.65).

6.2 Ergoregion

In the five-dimensional solutions studied in [31], one of the most striking and important properties of the solutions was that they have an ergoregion, where the timelike Killing vector at infinity becomes spacelike. The existence of an ergoregion is a characteristic property of the non-supersymmetric solutions: unbroken supersymmetry, by contrast, implies the existence of an everywhere causal Killing vector. In [39], this ergoregion was also shown to imply that the non-supersymmetric solutions of [31] were unstable, using a general argument due to [40]. It is therefore clearly important to study the ergoregion in our solutions.

It is difficult to analyse the ergoregion in the ten-dimensional geometry (3.53). The most general Killing vector which is timelike at large ρ is a linear combination of ∂_t, ∂_y, and ∂_z, $\xi = \partial_t - a\partial_y - b\partial_z$. We have

$$\xi \cdot \xi = \frac{1}{\sqrt{\tilde{H}_1 \tilde{H}_5}} \left[A(a - s_1 s_5 V_0 b)^2 - G + 2C c_1 c_5 b + D b^2 \right]. \tag{6.1}$$

Requiring this to be timelike at large ρ imposes

$$\left(a + s_1 s_5 \sqrt{\frac{q(q^2 + m^2)}{p(p^2 + m^2)}} b \right)^2 + b^2 < 1. \tag{6.2}$$

The expression for $\xi \cdot \xi$ is complicated. To get some insight, we can examine its behaviour in the corners: at $\rho = \rho_0, \theta = 0$,

$$\xi \cdot \xi = \frac{1}{\sqrt{\tilde{H}_1 \tilde{H}_5}} \left[4p\rho_0 n^2 (a - V_0 s_1 s_5 b)^2 - 4\rho_0 (p - \rho_0 (n^2 - 1)) \right], \tag{6.3}$$

where

$$V_0(\rho = \rho_0, \theta = 0) = -(n^2 - 1) \sqrt{\frac{\rho_0 (p + \rho_0)^3}{n^2 p (p - \rho_0 (n^2 - 1))}}. \tag{6.4}$$

At $\rho = \rho_0, \theta = \pi$,

$$\xi \cdot \xi = \frac{4\rho_0 (\rho_0 + p)^2 (n^2 - 1)}{(p - \rho_0 (n^2 - 1)) \sqrt{\tilde{H}_1 \tilde{H}_5}} [1 + (n c_1 c_5 - s_1 s_5) b][1 + (n c_1 c_5 + s_1 s_5) b]. \tag{6.5}$$

A necessary condition for ξ to be everywhere timelike in the ten-dimensional geometry is that we can choose a and b to make (6.3) and (6.5) negative while satisfying (6.2). We

have not analysed these conditions in detail; they depend in a complicated way on the charges.

Instead, we will study the ergoregion in the four-dimensional metric we obtain by Kaluza-Klein reduction. The ergoregion in the four-dimensional metric is in general different from the ergoregion in the ten-dimensional metric, since in the Kaluza-Klein reduction, we project the Killing vector down to four dimensions, losing the contribution to its norm from the first line in (3.62). The instability of [40, 39] was determined by the presence of an ergoregion in the asymptotically flat metric, so the ergoregion in the four-dimensional metric would seem to be more relevant to the question of stability. It also turns out to be much easier to determine. For this, we use (3.62), where the four-dimensional metric is given, up to a conformal factor, by the second line. But we have shown that $f > 0$, $A > 0$, and $D > 0$ away from the degenerations, so in this 4d metric, ∂_t is timelike everywhere. Thus, there is no ergoregion in the 4d metric!

This might seem quite surprising, but we can understand the difference from the five-dimensional case on general grounds, without detailed calculation. The four-dimensional metric we obtain upon Kaluza-Klein reduction is given by (3.62) for some D. Now for this to have an ergoregion, we would need g_{tt} to change sign while the four-dimensional metric remains of fixed signature. If we think of the second line of (3.62) as the t direction fibred over a three-dimensional base metric, to preserve the overall signature, the determinant of the base metric would have to change sign. But these terms are clearly all everywhere positive: in particular, the factor in front of $d\phi^2$ is positive away from the degenerations. The difference in the five-dimensional case was that we had a pair of angular directions, so the determinant of the four-dimensional base metric could change sign without encountering any degenerations. Thus, we expect the absence of the ergoregion in the four-dimensional solution to be a general property of such solutions.

Thus, the Killing vector $V = \partial_t$ is timelike everywhere in the four-dimensional spacetime. Assuming that we consider test fields propagating on this spacetime which satisfy the dominant energy condition, it follows that the energy constructed by integration over a Cauchy surface,

$$\mathcal{E} = \int_S V^\mu T_\mu{}^\nu dS_\nu, \tag{6.6}$$

will always be positive for any initial data. Hence, the instability discussed in [40, 39] cannot arise in this case. It is then an open question whether our non-supersymmetric solutions are unstable. There is no mechanism that would prevent them from being unstable, so past experience biases us to think that they will be, but this is a very interesting question for future research.

6.3 Near-core limit

The solutions we have constructed look qualitatively like smooth D1-D5 solutions sitting at the core of a Kaluza-Klein monopole. We would therefore expect to find that there is a suitable decoupling limit of the geometry in which we focus on the core region,

and obtain an $\mathrm{AdS}_3 \times S^3$ geometry. As in the previous non-supersymmetric case [31], obtaining such a limit will require us to scale some of the charges in a suitable way, going close to extremality. In this section, we will construct the decoupling limit for these solutions.

In the parametrization of section 4.2, the only free parameters are p, ρ_0 and the charge parameters. It seems natural to consider a limit where we take $\rho_0 \to 0$, while holding p and the physical D1 and D5 charges fixed: that is, we hold $\rho_0 \sinh \delta_i \cosh \delta_i$ fixed. Note that this is *n*ot the same as the extremal limit introduced in section 3.4, in which we took $m \to 0$ holding b fixed. In fact, such a limit is incompatible with the constraints imposed by the smoothness conditions. Thus, here we are not taking the extremal limit with all the charges held fixed; instead, we are scaling $\mathcal{Q}$ and $\mathcal{J}$ to zero.

As we take this limit, we scale the coordinates so as to zoom in on a 'core' region in the geometry, by setting $\rho = \rho_0 r$ and holding r fixed. As we take the limit, the identification on the y coordinate scales like $1/\sqrt{\rho_0}$. It is therefore convenient to set $y = \chi/4\sqrt{p\rho_0}$. It will also be convenient to set $t = \tau/4\sqrt{p\rho_0}$ and $z = p\psi$. In this limit, the metric (3.53) becomes

$$
\begin{aligned}
ds^2_{10} \approx & \frac{1}{4\ell^2}\left\{ a\left[d\chi + \frac{\ell^2 n}{2}\left(\frac{(1+\cos\theta)}{a}(d\psi + \bar{\omega}^1) + \bar{\kappa}^1 \right)\right]^2 \right. \\
& \left. - g\left[d\tau + \frac{\ell^2 n}{2}\left(\bar{\omega}^0 - \frac{(1-\cos\theta)}{g}(d\psi + \bar{\omega}^1)\right)\right]^2 \right\} \\
& + \frac{\ell^2}{4}\left[\frac{dr^2}{r^2-1} + d\theta^2 + \frac{r^2 - 1 + n^2\sin^2\theta}{ag}(d\psi + \bar{\omega}^1)^2 + \frac{(r^2-1)\sin^2\theta}{(r^2 - 1 + n^2 \sin^2\theta)} d\phi^2 \right] \\
& + \sqrt{\frac{\mathcal{Q}_1}{\mathcal{Q}_5}} ds^2_{T^4},
\end{aligned}
\tag{6.7}
$$

where we have set

$$
\ell^2 = 4\sqrt{\tilde{H}_1 \tilde{H}_5} = 16p\sqrt{\mathcal{Q}_1 \mathcal{Q}_5},
\tag{6.8}
$$

and

$$
a = 2(r - 1 + n^2(1+\cos\theta)), \quad g = 2(r + 1 - n^2(1-\cos\theta)),
\tag{6.9}
$$

$$
\bar{\omega}^0 = \frac{(r-1)\sin^2\theta}{r^2 - 1 + n^2\sin^2\theta} d\phi,
\tag{6.10}
$$

$$
\bar{\omega}^1 = 2\frac{(r^2-1)\cos\theta - n^2 r^2 \sin^2\theta}{r^2 - 1 + n^2 \sin^2\theta} d\phi,
\tag{6.11}
$$

$$
\bar{\kappa}^1 = \frac{(r+1)\sin^2\theta}{r^2 - 1 + n^2\sin^2\theta}.
\tag{6.12}
$$

This metric has an $\mathrm{AdS}_3 \times S^3$ geometry (at least locally). This can be made explicit by introducing new angular coordinates

$$
\bar{\psi} = \frac{1}{4}(2\phi + \psi), \quad \bar{\phi} = \frac{1}{4}(2\phi - \psi),
\tag{6.13}
$$

and writing

$$r = 1 + 2R^2, \quad \chi = \ell^2 \varphi, \quad \theta = 2\bar{\theta}. \tag{6.14}$$

In terms of these coordinates (6.7) becomes

$$\begin{aligned} ds^2 = &- \frac{(R^2+1)}{\ell^2} d\tau^2 + \frac{\ell^2 dR^2}{R^2+1} + \ell^2 R^2 d\varphi^2 \\ &+ \ell^2 (d\bar{\theta}^2 + \cos^2\bar{\theta}(d\bar{\psi} + n d\varphi)^2 + \sin^2\bar{\theta}(d\bar{\phi} - \frac{n}{\ell^2} d\tau)^2) + \sqrt{\frac{\mathcal{Q}_1}{\mathcal{Q}_5}} ds^2_{T^4}. \end{aligned} \tag{6.15}$$

The identifications (4.9, 4.10, 4.13) become in these coordinates simply $\bar{\psi} \sim \bar{\psi} + 2\pi$, $\bar{\phi} \sim \bar{\phi} + 2\pi$ and $(\varphi, \bar{\psi}) \sim (\varphi - 2\pi, \bar{\psi} + 2\pi n)$. These identifications make the spacetime globally $\mathrm{AdS}_3 \times S^3$. Recall however that these may not be the fundamental identifications. For example, if we adopt the basis of identifications (4.16), the geometry becomes $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_{N_K}$. More general choices will give other orbifolds of $\mathrm{AdS}_3 \times S^3$ in the decoupling limit.

The dilaton is a constant in this limit. For the form field, we first need to make a gauge transformation to get the correct behaviour in the limit: we shift $C^{(2)} \to C^{(2)} - dt \wedge dy$, so that

$$\begin{aligned} C^{(2)} &= c_5 s_5 \mathcal{C} + s_5 c_5 s_1^2 \frac{1-H}{H_1} \mathcal{B} \wedge \mathcal{A} + \frac{c_1 s_5 \mathcal{B}}{H_1} \wedge dt - \frac{s_1 c_5 \mathcal{A}(1-H)}{H_1} \wedge dy \\ &\quad - \frac{(1 - s_1(c_1 - s_1)H)}{H_1} dt \wedge dy. \end{aligned} \tag{6.16}$$

Now as $\rho_0 \to 0$, this will become

$$\begin{aligned} C^{(2)} =& p\mathcal{Q}_5 \left(\frac{\mathcal{C}}{p\rho_0(n^2-1)} + \frac{16 n^2 \sin^2\theta}{a} d\bar{\psi} \wedge d\bar{\phi} \right) + \sqrt{\frac{\mathcal{Q}_5}{\mathcal{Q}_1}} \frac{n}{2} (1 + \cos\theta) d\bar{\psi} \wedge d\tau \\ &- \sqrt{\frac{\mathcal{Q}_5}{\mathcal{Q}_1}} \frac{n}{2} (1 - \cos\theta) d\bar{\phi} \wedge d\chi - \frac{1}{32 p \mathcal{Q}_1} (r + 1 - n^2(1 - \cos\theta)) d\tau \wedge d\chi. \end{aligned} \tag{6.17}$$

In the limit,

$$\mathcal{C} = dx^5 \wedge (-V_t \omega^0 + \kappa_t^0) \approx -16\rho_0 p (n^2-1) \frac{((r-1)\cos\theta + n^2(1+\cos\theta))}{a} d\bar{\psi} \wedge d\bar{\phi}, \tag{6.18}$$

so discarding some pure gauge terms from (6.17), the two-form becomes

$$C^{(2)} = -8p\mathcal{Q}_5 \left[\cos\theta (d\bar{\psi} + \frac{n}{\ell^2} d\chi) \wedge (d\bar{\phi} - \frac{n}{\ell^2} d\tau) + \frac{1}{\ell^4} r d\tau \wedge d\chi \right], \tag{6.19}$$

which is of the expected form to correspond to an $\mathrm{AdS}_3 \times S^3$ solution.

6.4 Four-dimensional description

Finally, we will make a brief remark about the structure of the four-dimensional metric obtained by Kaluza-Klein reduction. The four-dimensional metric in the Einstein frame is

$$ds_4^2 = -\frac{f^2}{\sqrt{AD}}(dt + c_1 c_5 \omega^0)^2 + \sqrt{AD}\left[\frac{d\rho^2}{\Delta} + d\theta^2 + \frac{\Delta}{f^2}\sin^2\theta d\phi^2\right]. \tag{6.20}$$

We can think of this as a fibration over the three-dimensional base metric

$$ds_3^2 = \frac{d\rho^2}{\rho^2 - \rho_0^2} + d\theta^2 + \frac{(\rho^2 - \rho_0^2)\sin^2\theta}{(\rho^2 - \rho_0^2) + n^2\rho_0^2\sin^2\theta} d\phi^2. \tag{6.21}$$

This is exactly the same base metric found in eq. (3.22) of [32] (with $(m-n)_{there} = n_{here}$). Thus, passing from the five-dimensional solutions described there to the four-dimensional one we consider modifies only the fibration, and not the three-dimensional base metric, which is what we would expect when adding a Kaluza-Klein monopole charge.

As a result, the structure of the four-dimensional metric is the same as in [32]. In particular, while the four-dimensional metric is smooth at $\theta = 0, \pi$, there is a $\mathbb{Z}_n$ orbifold singularity at $\rho = \rho_0$, and there are curvature singularities in the three-dimensional base metric at the corners $\rho = \rho_0$, $\theta = 0, \pi$. These curvature singularities in the base metric do not have simple brane interpretation. Hence, as in [32], the smooth solutions we have found here do not fit into the picture of [27], where supersymmetric solutions were described as built up from half-BPS atoms.

Acknowledgments

We would like to thank Jon Ford, Eric Gimon, Tommy Levi, Samir Mathur and Yogesh Srivastava for valuable discussions and correspondence. The authors would like to acknowledge the hospitality of the Aspen Center for Physics during the initial stages of this work. SG and AS were supported by NSERC. SFR was supported in part by PPARC.

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